ISRAEL JOURNAL OF MATHEMATICS 87 (1994), 361–401

SPLITTING METAPLECTIC COVERS OF DUAL REDUCTIVE PAIRS

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ABSTRACT

The symplectic group $\mathrm{Sp}(n,F)$ over a local field F (other than $\mathbb{C}$) has a unique non-trivial twofold central extension. The inclusion of $\{\pm 1\}$ into the circle $\mathbb{C}^1$ induces an extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(n,F) \longrightarrow \mathrm{Sp}(n,F) \longrightarrow 1.$$

In this paper, an explicit splitting of the restriction of this extension to a dual reductive pair (G,H) in $\mathrm{Sp}(n,F)$ is given in all cases in which it exists. Such an explicit splitting is often an essential technical ingredient in the study of the local theta correspondence for the dual pair [4].

Introduction

The cocycle of the metaplectic cover of the symplectic group over a local field has been well studied and has a nice expression in terms of the Leray invariant [8], [6]. On the other hand, in many applications it is the restriction of the metaplectic cover to a dual reductive pair which is of primary concern. While it is known that this restriction frequently splits [5], Chapt. 3., a nice explicit formula for the splitting does not seem to occur in the literature. In this strictly utilitarian note we give such a splitting in all cases in which it exists.

More precisely, suppose that F is complete local field of characteristic zero. Let $\mathbb{W}$ denote a symplectic vector space of dimension $2n$ over F, and let $\mathrm{Sp}(\mathbb{W})$

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* Partially supported by NSF Grant DMS-9003109.
Received June 17, 1993

be the corresponding symplectic group of rank n over F. We view Sp(W) as acting on W on the right.

If $F \neq \mathbb{C}$, the group Sp(W) has a non-trivial 2-fold cover

$$1 \longrightarrow \mu_2 \longrightarrow \tilde{\mathrm{Sp}}(\mathbf{W}) \longrightarrow \mathrm{Sp}(\mathbf{W}) \longrightarrow 1, \tag{0.1}$$

where μ_r denotes the r-th roots of unity. For many purposes it is more convenient to work with the associated $\mathbb{C}^1$-extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(\mathbf{W}) \longrightarrow \mathrm{Sp}(\mathbf{W}) \longrightarrow 1, \tag{0.2}$$

obtained by including μ_2 into $\mathbb{C}^1$. It should also be noted that the $\mathbb{C}^1$-extension used here splits over certain subgroups for which the corresponding μ_2 extension does not split.

A cocycle for this extension has a very simple expression [8], [6], [5] in terms of the Leray invariant. Let $\Omega = \Omega(\mathbf{W})$ be the space of n planes in W which are isotropic for $\langle , \rangle$. The symplectic group acts transitively on Ω and on the set of pairs $U_1, U_2 \in \Omega$ which are transverse ($U_1 \cap U_2 = 0$). To a given ordered triple $U_1, U_2, U_3 \in \Omega$ which are pairwise transverse there is associated an n-dimensional F vector space $L = L(U_1, U_2, U_3)$ with a symmetric, non-degenerate, F-bilinear form $(\, , \,)_L$. The isometry class of this form is uniquely defined and is the Leray invariant of the triple.

The Leray invariant depends only on the Sp(W) orbit of such a pairwise transverse triple and the orbits of Sp(W) on the set of such triples are characterized by their Leray invariant. If $U_1, U_2, U_3 \in \Omega$ is an arbitrary triple, let

$$R = (U_1 \cap U_2) + (U_2 \cap U_3) + (U_3 \cap U_1) \tag{0.3}$$

and let $\mathbf{W}_R = R^{\perp}/R$ with the natural symplectic form induced by $\langle , \rangle$. For any $U \in \Omega$, let U_R be the image of $U \cap R^{\perp}$ in $\mathbf{W}_R$. Then the triple $U_{1,R}$, $U_{2,R}$, $U_{3,R} \in \Omega(\mathbf{W}_R)$ is pairwise transverse and one defines

$$L(U_1, U_2, U_3) := L(U_{1,R}, U_{2,R}, U_{3,R}), \tag{0.4}$$

so that $L(U_1, U_2, U_3)$ is an isometry class of symmetric bilinear forms of dimension $n - r$,where $r = \dim R$. Again the Sp(W) orbit of the triple U_1, U_2, U_3 is determined by the dimensions of their mutual intersections and their Leray invariant [8], Theorem 2.11. All of these facts are very nicely described in Rao's paper [8] .

Finally, we fix a non-degenerate additive character ψ of F, and, for any (non-degenerate) symmetric F-bilinear form L, we let $\gamma_F(\psi \circ L) \in \mu_8$ denote the Weil invariant of the character of second degree $x \mapsto \psi((x,x)_L)$.

Remark: At this point there is a choice to be made. If $(\ ,\ )_L\colon L \times L \longrightarrow F$ is a symmetric bilinear form, we could have taken

$$L[x] = \frac{1}{2}(x,x)_L \tag{0.5}$$

as the associated quadratic form, following a classical convention, as, say, in Cassels [2], p .7, in introducing the factor of $\frac{1}{2}$. For example, if $L = F$ with $(x,y) = xy$, then $L[x] = \frac{1}{2}x^2$. On the other hand, in our definition of $\gamma_F(\psi \circ L)$, we have followed Rao's convention in taking the quadratic form $x \mapsto (x,x)_L = 2L[x]$. This was done to maintain consistency with his formulas. The price, however, is that in many places it is the Weil index $\gamma(\frac{1}{2}\psi \circ L)$ which occurs, i.e., the additive character ψ is replaced by $\eta = \frac{1}{2}\psi$, $\eta(x) = \psi(\frac{1}{2}x)$. On the positive side, the matrix $((x_i,x_j))$ of inner products for a basis $\{x_i\}$ for L is the matrix both for the inner product and for the quadratic form $x \mapsto (x,x)_L$ which is used in the calculation of $\gamma(\psi \circ L)$.

The result of Rao and Perrin is the following:

THEOREM (([8], [6])): *For any fixed $\mathbb{Y} \in \Omega(\mathbb{W})$ there is an isomorphism*

$$\mathrm{Mp}(\mathbb{W}) \simeq \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^1$$

where

$$(g_1,\epsilon_1)(g_2,\epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 c_{\mathbb{Y}}(g_1,g_2))$$

with cocycle $c_{\mathbb{Y}}$ given by

$$c_{\mathbb{Y}}(g_1,g_2) = \gamma_F(\frac{1}{2}\psi \circ L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)).$$

Remark: The second statement here should be compared with Theorem 4.1 (5), p. 358 of [8].

Since the Weil invariant is identically 1 when $F = \mathbb{C}$, we will exclude this case from now on.

Our goal is to describe the restriction of the metaplectic cover to certain dual reductive pairs (G, H) in $\mathrm{Sp}(\mathbf{W})$. These are defined as follows: Let D be a division algebra whose center E contains F, and let $\tau\colon D \longrightarrow D$ be an involution of D (anti-automorphism of D of order 2). Assume that F is the set of fixed points of the restriction of τ to E. The possibilities for (D, E, F, τ) are

1. $D = E = F$ and $\tau = \mathrm{id}$.
2. D is the unique division quaternion algebra over $E = F$ and τ is the main involution of D.
3. $D = E$ is a quadratic extension of F and τ is the non-trivial Galois automorphism.

Fix $\epsilon = \pm 1$. Let $W \simeq D^{2n}$ (row vectors) be a left vector space over D of dimension $2n$ with ϵ-skew hermitian form

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 \cdot {}^t y_2^\tau - \epsilon\, y_1 \cdot {}^t x_2^\tau, \tag{0.6}$$

and let

$$G = \{ g \in GL_{2n}(D) \mid \quad \langle w_1 g, w_2 g \rangle = \langle w_1, w_2 \rangle,\ \forall w_1, w_2 \in W \ \}, \tag{0.7}$$

be the isometry group of W. Note that W has a complete polarization (decomposition as a direct sum of a pair of maximal isotropic subspaces) $W = X + Y$ where $X = \{(x, 0) |\ x \in D^n\}$ and $Y = \{(0, y) |\ y \in D^n\}$. Let V, $(\ ,\ )$ be a right D vector space of dimension m with a non-degenerate ϵ-Hermitian form and let H denote the isometry group of V. If we let $d = \dim_F D$ and let $\mathrm{tr}\colon D \longrightarrow F$ be the reduced trace, then

$$\mathbf{W} = V \otimes_D W, \qquad \langle\langle\ ,\ \rangle\rangle = \kappa \cdot \mathrm{tr}\Big((\ ,\ ) \otimes \langle\ ,\ \rangle^\tau \Big), \tag{0.8}$$

where

$$\kappa = \begin{cases} 1 & \text{in case 1} \\ \frac{1}{2} & \text{in cases 2 and 3,} \end{cases} \tag{0.9}$$

is a symplectic vector space over F of dimension $2nmd$. There is a natural homomorphism

$$\iota\colon G \times H \longrightarrow \mathrm{Sp}(\mathbf{W}) \tag{0.10}$$

and (G, H) is a dual reductive pair of type I in $\mathrm{Sp}(\mathbf{W})$. It is not the most general such pair, since we have assumed that W is even dimensional over D and split i.e., admits a complete polarization. There are 6 cases, 1_ϵ, 2_ϵ and 3_ϵ. Of course, Hermitian and skew-Hermitian forms may be identified in case 3, but this identification is not canonical, so we prefer to avoid it.

Let $\Omega(W)$ denote the space of D subspaces $U \subset W$, with $\dim_D U = n$ which are isotropic for $\langle , \rangle$. Again G acts transitively on $\Omega(W)$ and on the set of pairs U_1 $U_2 \in \Omega(W)$ which are transverse. Moreover, given a triple U_1, U_2, and $U_3 \in \Omega(W)$, the definition of the Leray invariant can be carried over without essential change – see §1– to yield an invariant $L_D(U_1, U_2, U_3)$ which is an ϵ-Hermitian form over D of rank $n - r$ ($r = \dim_D R$, as before) and which is uniquely determined up to isometry. Here, to be precise, we should view $L = L_D(U_1, U_2, U_3)$ as a left D vector space with an ϵ-Hermitian form $(\, , \,)_L \colon L \times L \longrightarrow D$. Moreover, for a fixed V as above, there is a map

$$(0.11) \qquad \mu_V \colon \left\{ \begin{matrix} \epsilon\text{-Hermitian forms over } D \\ \text{of rank } k \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{symmetric forms over } F \\ \text{of rank } mkd \end{matrix} \right\}$$

given by

$$(0.12) \qquad L \mapsto V \otimes_D L \qquad (\, , \,)_{\mu_V(L)} = \kappa \cdot tr\Big((\, , \,)_V \otimes (\, , \,)_L^{\tau} \Big).$$

Finally, there is a natural map

$$(0.13) \qquad R_V \colon \Omega(W) \longrightarrow \Omega(\mathbf{W})$$

given by

$$(0.14) \qquad R_V U = V \otimes_D U.$$

Let $\iota_V \colon G \longrightarrow \mathrm{Sp}(\mathbf{W})$ and $\iota_W \colon H \longrightarrow \mathrm{Sp}(\mathbf{W})$ denote the restrictions of ι to $G \times 1$ and $1 \times H$ respectively. Our first observation is the following:

Proposition 0.1:

(i) *R_V is compatible with the Leray invariant, i.e.,*

$$\mu_V(L_D(U_1, U_2, U_3)) = L(R_V U_1, R_V U_2, R_V U_3).$$

Thus the diagram

$$\begin{array}{ccc} \Omega(W)^3 & \xrightarrow{\;R_V\;} & \Omega(\mathbb{W})^3 \\ \big\downarrow{\scriptstyle L_D} & & \big\downarrow{\scriptstyle L} \\ Herm^\epsilon(D)/\sim & \xrightarrow{\;\mu_V\;} & Sym(F)/\sim, \end{array}$$

commutes, where $\sim$ denotes equivalence up to isometry.

(ii) *In particular, for any $Y \in \Omega(W)$, let $\mathbb{Y} = R_V Y$. Then*

$$c_{\mathbb{Y}}(\iota_V(g_1), \iota_V(g_2)) = \gamma_F(\,\frac{1}{2}\psi \circ \mu_V(L_D(Y, Yg_2^{-1}, Yg_1))\,).$$

Thus the Leray invariant and the Perrin-Rao cocycle are 'natural'.

On the other hand, the isometry classes of symmetric bilinear forms over F are characterized by their dimension, determinant and Hasse invariant in the non-archimedean case, and by their dimension and signature in archimedean case [10]. A similar, and often simpler, set of invariants characterize the isometry classes of ϵ-Hermitian forms over D, [7], [9], and the map μ_V has an explicit description in terms of these invariants (Proposition 2.1). Moreover, the Weil index $\gamma(\eta \circ L)$ of an arbitrary quadratic form L depends only on the isometry class of L and is determined in an explicit way by the invariants of L (Lemma 3.4). Using this description and that of μ_V, we calculate the cocycle $c_{\mathbb{Y}}(\iota_V(g_1), \iota_V(g_2))$. The fact that the invariants of L_D are usually rather simple allows us to find (Theorem 3.1) an explicit and relatively simple function β_V whose coboundary is $c_{\mathbb{Y}}$, except, of course, in case 1_+ with m odd, when the cocycle remains nontrivial.

In section 4, we consider the case in which the space W is not split. Here a doubling procedure suggested by Michael Harris is used to reduce this case to the split case and to give "explicit" splittings (Propositions 4.1, 4.6, and 4.8). Finally, in the last section, we again assume that W is split, and we describe the operators which give the action of $G(W)$ in the standard Schrödinger model of the Weil representation.

Section 1. The Leray invariant L_D and proof of Proposition 0.1

Section 2. Classification of elements of $Herm_n^\epsilon(D)$ and the map μ_V

Section 3. Explicit trivializations

Section 4. Other unitary groups, some examples

Section 5. Schrödinger models

ACKNOWLEDGEMENT: I would like to thank J. Adams for his patient and detailed reading of a rough first version of this paper. His comments greatly improved the exposition. I would also like to thank M. Harris for his, as always, fruitful suggestion concerning the splitting in the case of arbitrary unitary groups.

1. The Leray invariant L_D

In this section we sketch the definition of the generalized Leray invariant L_D. In fact the proofs in Rao's paper [8] go over with only minor changes, and so we simply will recall, usually without proof, some of the main structural facts which must be checked.

1.1 Let W, $\langle,\rangle$ be a split ϵ-skew Hermitian space over D and define G, $\Omega(W)$, etc. as in the introduction. For any $U \in \Omega(W)$, let $P_U \subset G$ be the stabilizer of U and let $N_U = \{g \in P_U | \, g|_U = \mathrm{id}\}$ be its unipotent radical.

Given a triple $U_1, U_2, U_3 \in \Omega(W)$, there exists an element $g \in N_{U_1}$ such that $U_2 g = U_3$ if and only if $U_2 \cap U_1 = U_3 \cap U_1$, and this element is unique if U_2 and U_3 are both transverse to U_1. This is Lemma 2.3 in [8]. Now suppose that U_1, U_2,and U_3 are pairwise transverse. Let $g \in N_{U_1}$ be the unique element such that $U_2 g = U_3$, and write

$$g = \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix} \qquad \rho \in \mathrm{Hom}_D(U_2, U_1) \tag{1.1}$$

with respect to the complete polarization $W = U_2 + U_1$. Then let

$$L_D = L_D(U_1, U_2, U_3) = U_2$$

and for x and $y \in L_D$, let

$$(x, y)_{L_D} = \langle x, y\rho \rangle = \langle x, yg \rangle. \tag{1.2}$$

Note that

$$\begin{aligned}(x, y)_{L_D} &= \langle x, y\rho \rangle \\ &= \langle x, yg - y \rangle \\ &= \langle xg, (yg - y)g \rangle \\ &= \langle xg, yg - y \rangle \\ &= -\langle xg, y \rangle \\ &= \epsilon \langle y, xg \rangle^{\tau} \\ &= \epsilon (y, x)_{L_D}^{\tau}.\end{aligned} \tag{1.3}$$

Thus L_D, $(\, , \,)_{L_D}$ is an ϵ-Hermitian space over D. It is non-degenerate:

$$\begin{aligned} (x,y)_{L_D} &= \langle x, y\cdot g\rangle = 0, \quad \forall x \in L_D = U_2 \\ \iff \quad & y\cdot g \in U_2^{\perp} \cap U_3 = U_2 \cap U_3 = 0, \end{aligned} \tag{1.4}$$

since U_2 and U_3 are transverse.

LEMMA 1.1: *Given ordered triples U_1, U_2, U_3 and $U_1', U_2', U_3' \in \Omega(W)$, each of which is pairwise transverse, there exists an $h \in G$ such that $U_i h = U_i'$ if and only if $L_D(U_1, U_2, U_3)$ and $L_D(U_1', U_2', U_3')$ are isometric.*

Proof: If an isometry

$$\sigma : L_D(U_1, U_2, U_3) = U_2 \xrightarrow{\sim} L_D(U_1', U_2', U_3') = U_2' \tag{1.5}$$

exists, then there is a unique extension of σ to an element $\sigma \in G$ such that $U_1\sigma = U_1'$. Let $g \in N_{U_1}$ and $g' \in N_{U_1'}$ be the elements used in defining L_D, i.e., such that $U_2 g = U_3$ and $U_2' g' = U_3'$. For x and $y \in U_2$, $x\cdot g\cdot\sigma$ is an arbitrary element of $U_3\sigma$ and $y\cdot\sigma\cdot g'$ is a arbitrary element of U_3'. We compute:

$$\begin{aligned} \langle x\cdot g\cdot\sigma, y\cdot\sigma\cdot g'\rangle &= \langle (x + x\cdot\rho)\cdot\sigma, y\sigma + y\sigma\rho'\rangle \\ &= \langle x\sigma, y\sigma\rangle + \langle x\sigma, y\sigma\rho'\rangle + \langle x\rho\sigma, y\sigma\rangle + \langle x\rho\sigma, x\sigma\rho'\rangle \\ &= 0 + \langle x, y\rho\rangle + \langle x\rho, y\rangle + 0 \\ &= (x,y)_{L_D} - \epsilon\langle y, x\rho\rangle^{\tau} \\ &= (x,y)_{L_D} - \epsilon(y,x)_{L_D}^{\tau} = 0. \end{aligned} \tag{1.6}$$

Thus $U_3\sigma \subset (U_3')^{\perp} = U_3'$, and hence the two must coincide. The other assertion is easy. ∎

Rao's proof that $L_D(U_1, U_2, U_3)$ is alternating for permutations of the U_i's carries over without change.

For an arbitrary triple U_1, U_2, $U_3 \in \Omega(W)$ we have

LEMMA 1.2 (Rao's 5-term Lemma): *There exists an orthogonal decomposition*

$$W = W_0 + W_1 + W_2 + W_3 + W_4$$

with W_i, $\langle , \rangle|_{W_i}$ non-degenerate and split, such that

$$U_i = W_0 \cap U_i + W_1 \cap U_i + W_2 \cap U_i + W_3 \cap U_i + W_4 \cap U_i$$

for $i = 1, 2, 3$ *and*

$$(0) \qquad U_1 \cap W_0 = U_2 \cap W_0 = U_3 \cap W_0$$

$$(1) \qquad U_1 \cap W_1 \textit{ is transverse to } U_2 \cap W_1 = U_3 \cap W_1$$

$$(2) \qquad U_2 \cap W_2 \textit{ is transverse to } U_3 \cap W_2 = U_1 \cap W_2$$

$$(3) \qquad U_3 \cap W_3 \textit{ is transverse to } U_1 \cap W_3 = U_2 \cap W_3$$

and

$$(4) \qquad U_1 \cap W_4, \quad U_2 \cap W_4, \quad U_3 \cap W_4 \quad \textit{are pairwise transverse.}$$

For later convenience we set $2r_i = \dim_D W_i$.

Now, as in Rao's case, set

$$R = U_1 \cap U_2 + U_2 \cap U_3 + U_3 \cap U_1 \tag{1.7}$$

and let $W_R = R^\perp/R$. Let $U_{i,R}$ denote the image of $U_i \cap R^\perp$ in W_R. Then define

$$L_D(U_1, U_2, U_3) = L_D(U_{1,R}, U_{2,R}, U_{3,R}). \tag{1.8}$$

Note that, if $r = \dim_D R$, then $\dim_D W_R = 2n - 2r$ and $L_D(U_1, U_2, U_3)$ is an ϵ-Hermitian space over D of dimension $\ell = n - r$. It is then easily checked using the five term Lemma that this Leray invariant together with the dimensions of the mutual intersections characterizes the G orbit of the triple.

Remark: Note that, while the above facts are almost entirely trivial extensions of the corresponding assertions in Rao, there are certain differences which must be kept in mind. For example, not all non-degenerate subspaces of W need be split in general. Moreover, in case 1_-, i.e., for W with a split symmetric bilinear form over F with $\dim_F W = 2n$ and n *odd, no pairwise transverse triples* U_1, U_2, $U_3 \in \Omega(W)$ *exist (!)* (the Leray invariant must be a *non-degenerate* alternating form of dimension n in this case) and the 'reduction' step is always necessary.

1.2 The facts about G orbits in $\Omega(W)$ just described yield structural information about G. To make this precise, again following Rao, let $e_1, \ldots, e_n, e'_1, \ldots, e'_n$ be the D basis for W which gives the isomorphism $W \simeq D^{2n}$ of the introduction, and define $\tau \in G$ by $e_i \cdot \tau = -\epsilon e'_i$ and $e'_i \cdot \tau = e_i$ for all i. Then $\tau^2 = -\epsilon \cdot 1_{2n}$. For any subset $S \subset \{1, \ldots, n\}$, define $\tau_S \in G$ by

$$e_i \cdot \tau_S = \begin{cases} e_i \cdot \tau & \text{if } i \in S \\ e_i & \text{if } i \notin S \end{cases} \tag{1.9}$$

$$e'_i \cdot \tau_S = \begin{cases} e'_i \cdot \tau & \text{if } i \in S \\ e'_i & \text{if } i \notin S. \end{cases} \tag{1.10}$$

Then it is easily checked that, for $Y = \mathrm{span}_D\{e'_1, \ldots, e'_n\}$ and for $P = P_Y$ and $N = N_Y$,

$$G = \coprod_{j=0}^{n} P\tau_j P \tag{1.11}$$

where $\tau_j = \tau_{\{1,\ldots,j\}}$, and $P\tau_S P = P\tau_j P$ if and only if $|S| = j$. If we write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to the complete polarization $W = X + Y$ defined in the introduction, then $\ker(c) = Y \cap Yg^{-1}$, and $g \in P\tau_j P$ where $j = n - \dim_D \ker(c)$.

Next, using the five term Lemma, we obtain:

PROPOSITION 1.3 (Rao): *Given g_1 and $g_2 \in G$, there exist p, p_1 $p_2 \in P$ such that*

$$g_1 = p_1\kappa_1 p^{-1}, \qquad g_2 = p\kappa_2 p_2$$

with

$$\kappa_1 = \mathrm{diag}[1, \tau, 1, \tau, \tau n(\rho)]$$
$$\kappa_2 = \mathrm{diag}[1, \tau, \tau, 1, \tau].$$

Here the block sizes in κ_1 and κ_2 are given by $2r_0, \ldots, 2r_4$, with $2r_j = \dim_D W_j$, $j = 0, 1, 2, 3, 4$ where $W = W_0 + W_1 + \cdots + W_4$ is the five term decomposition associated to the triple Y, Yg_2^{-1}, Yg_1. Moreover, $\rho = \epsilon\rho^\tau$ is a nondegenerate ϵ-Hermitian matrix in the isometry class of $L_D(Y, Yg_2^{-1}, Yg_1)$.

Here, if $\rho \in \mathrm{Herm}_n^\epsilon(D)$ we write

$$n(\rho) = \begin{pmatrix} 1_n & \rho \\ 0 & 1_n \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 0 & -\epsilon \cdot 1_n \\ 1_n & 0 \end{pmatrix}, \tag{1.12}$$

and similarly in the smaller blocks.

We will need to carry over Rao's function $x(g)$. If $g = p_1\tau_S p_2 \in P\tau_S P$, then

$$p_1 p_2|_Y \in GL_D(Y) \simeq GL_n(D), \tag{1.13}$$

and we let

$$x(g) = \nu(p_1p_2|_Y) \in E^\times \tag{1.14}$$

be the reduced norm of this element. To determine the extent to which $x(g)$ depends on the choice of p_1, p_2 and S we need the following (which is just a translation of Rao's consistency Lemma 3.5 to our context):

LEMMA 1.4: *Suppose that p_1 and $p_2 \in P$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ are such that $p_1 g p_2 = g$. Let $\bar{Y} = Y/\ker(c)$. Then $p_1|_Y$ preserves $\ker(c) = Y \cap Yg^{-1}$ and, if $\bar{p}_1$ denotes the endomorphism of $\bar{Y}$ induced by p_1, then*

$$(\det(\bar{p}_1))(\det(\bar{p}_1))^\tau = \det(p_1p_2|_Y).$$

Proof: As in Rao we can reduce to the case $g = \tau_S$ and we write everything with respect to the decomposition $W = X_{S'} + X_S + Y_S + Y_{S'}$, e.g.,

$$\tau_S = \begin{pmatrix} 1 & & & \\ & 0 & -\epsilon & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}. \tag{1.15}$$

Then $p_1\tau_S p_2 = \tau_S$ implies that p_1 and $p_2 \in P \cap P_{Y\tau_S}$. Since $Y\tau_S = X_S + Y_{S'}$, we have $Y_{S'} = Y\tau_S \cap Y$ is preserved by p_1 and p_2. This shows that p_1 and p_2 preserve the flag $W \supset X_S + Y \supset Y \supset Y_{S'}$, and hence are upper triangular. Write $p_1 = (a_{ij})$ and $p_2^{-1} = (b_{ij})$. The condition $p_1\tau_S = \tau_S p_2^{-1}$ then implies that $a_{21} = a_{43} = 0$, $b_{21} = b_{43} = 0$ and

$$a_{11} = b_{11}, \qquad a_{22} = b_{33}, \qquad a_{33} = b_{22}, \qquad a_{44} = b_{44}, \tag{1.16}$$

and $a_{23} = 0$. Thus

$$\begin{aligned} \det(p_1p_2|_Y) &= \det(p_1|_Y)\det(p_2^{-1}|_Y)^{-1} \\ &= \det(a_{33})\det(a_{44})\det(b_{33})^{-1}\det(b_{44})^{-1} \\ &= \det(a_{33})\det(a_{22})^{-1}. \end{aligned} \tag{1.17}$$

But now, on the $W_S = X_S + Y_S$ block, i.e., on $X_S + Y/Y_{S'}$, p_1 has matrix

$$\begin{pmatrix} a_{22} & 0 \\ 0 & a_{33} \end{pmatrix} = \begin{pmatrix} ({}^t a_{33}^\tau)^{-1} & 0 \\ 0 & a_{33} \end{pmatrix}. \tag{1.18}$$

so that $a_{22} = ({}^t a_{33}^\tau)^{-1}$, and $\det(a_{22})^{-1} = \det(a_{33})^\tau$. But finally, $Y \cap Y\tau_S^{-1} = Y_{S'}$ so that $\bar{Y} = Y/Y_{S'} \simeq Y_S$, and $\det(\bar{p}_1) = \det(a_{33})$. This proves the relation we want. ∎

COROLLARY 1.5: *The quantity $x(g)$ is a well defined element of $F^\times/F^{\times,2}$ in cases 1 and 2, and of $E^\times/NE^\times$ in case 3.*

Note that the quantity $x(g)$ actually depends on the choice of standard basis $e_1, \dots, e'_n$.

Finally, the following relation, which occurs in Rao's paper, will be of fundamental importance:

PROPOSITION 1.6: *Suppose that g_1 and $g_2 \in G$ with $g_1 \in P\tau_{j_1}P$, $g_2 \in P\tau_{j_2}P$ and $g_1g_2 \in P\tau_jP$. Let $\rho = L_D(Y, Yg_2^{-1}, Yg_1)$ be the associated Leray invariant and set $\ell = \dim_D L_D(Y, Yg_2^{-1}, Yg_1) = \dim_D \rho$. Define t by the relation*

$$2t = j_1 + j_2 - j - \ell.$$

Then

$$x(g_1g_2) = x(g_1)x(g_2)(-\epsilon)^t \det(\rho)$$

where $\det(\rho)$ is to be interpreted as the reduced norm of $\rho \in M_\ell(D)$ in case 2.

Proof: Consider the decompositions $g_1 = p_1\kappa_1p^{-1}$,and $g_2 = p\kappa_2p_2$ as in Proposition 1.3 and note that

$$\begin{aligned} j_1 &= r_1 + r_3 + \ell \\ j_2 &= r_1 + r_2 + \ell \\ j &= r_2 + r_3 + \ell. \end{aligned} \tag{1.19}$$

We have

$$\begin{aligned} x(g_1) &= x(p_1)x(p)^{-1} \\ x(g_2) &= x(p)x(p_2) \\ x(g_1g_2) &= x(p_1)x(\kappa_1\kappa_2)x(p_2). \end{aligned} \tag{1.20}$$

Now

$$\kappa_1\kappa_2 = \mathrm{diag}[1, -\epsilon, \tau, \tau, \tau n(\rho)\tau] \tag{1.21}$$

and

$$\begin{aligned} \tau n(\rho)\tau &= \begin{pmatrix} -\epsilon & 0 \\ \rho & -\epsilon \end{pmatrix} \\ &= \begin{pmatrix} ({}^t\rho^{-1})^\tau & 0 \\ 0 & \rho \end{pmatrix} n(-\epsilon^t\rho^\tau)\tau n(-\epsilon\rho^{-1}). \end{aligned} \tag{1.22}$$

Note that the block sizes in κ_i are r_0, r_1, r_2, r_3, and ℓ in our present notation. Since

$$j_1 + j_2 - j = 2r_1 + \ell \tag{1.23}$$

we conclude that $r_1 = t$ and that $x(\kappa_1\kappa_2) = (-\epsilon)^{r_1} \det \rho$, as required. ∎

1.3

Proof of Proposition 0.1: We now fix V, W, $\mathbf{W}$ etc. as in the introduction. For a triple U_1,U_2,$U_3 \in \Omega(W)$, which is pairwise transverse, let R_VU_1, R_VU_2, $R_VU_3 \in \Omega(\mathbf{W})$ be the corresponding triple. Let $L_D = L_D(U_1, U_2, U_3)$ and $L = L(R_VU_1, R_VU_2, R_VU_3)$ be the Leray invariants. Note that if $g \in N_{U_1}$ is the unique element such that $U_2g = U_3$, then $\iota_V(g) \in N_{R_VU_1}$ is likewise the unique element such that $R_VU_2\iota_V(g) = R_VU_3$. Then, for x and $y \in U_2$ and v and $w \in V$, we have

$$\begin{aligned}(v\otimes x, w\otimes y)_L &= \langle\langle v\otimes x, w\otimes (yg)\rangle\rangle \\ &= \kappa\cdot tr((v,w)\langle x, yg\rangle^{\tau}) \\ &= \kappa\cdot tr((v,w)(x,y)^{\tau}_{L_D}) \\ &= (v\otimes x, w\otimes y)_{\mu_V(L_D)}.\end{aligned} \tag{1.24}$$

This proves (i) of Proposition 0.1 in the pairwise transverse case. In general, it is clear that the 'reduction' procedure is compatible with R_V, and so (i) of Proposition 0.1 is proved. Part (ii) then follows immediately. ∎

2. Classification of ϵ-Hermitian forms over D and the map μ_V

In this section we first recall the invariants which characterize an ϵ-Hermitian form over D in each of the cases 1_ϵ, 2_ϵ and 3_ϵ. Good references for this material are [7] and [9]. We then describe the map μ_V: $\mathrm{Herm}^\epsilon(D) \longrightarrow \mathrm{Sym}(F)$ in terms of these invariants and those of V.

2.1 Let L be an ϵ-Hermitian space over D. Recall that in the three cases we have: 1. $D = E = F$, $\tau = \mathrm{id}$, 2. D quaternion algebra over $E = F$, $\tau =$ main involution, and 3. $D = E$, quadratic extension of F, τ the non-trivial Galois automorphism. Then invariants which characterize the isometry class of L are given by the following table:

Table 2.1

Invariants of ϵ-Hermitian forms (F non-archimedean)

Case	invariants		
1_+ L symmetric bilinear	$\dim_F L$	$\det L \in F^\times/F^{\times,2}$	$h_F(L)$
1_- L symplectic	$\dim_F L$		
2_+ L quaternion Hermitian	$\dim_D L$		
2_- L quaternion skew-Hermitian	$\dim_D L$	$\det L \in F^\times/F^{\times,2}$	
3_+ L Hermitian	$\dim_E L$	$\det L \in F^\times/NE^\times$	
3_- L skew-Hermitian	$\dim_E L$	$\det L \in (\delta)^{\dim_E L} F^\times/NE^\times$	

Here, for example, in case 1_+ we choose a basis $\{x_i\}$ for the F vector space L such that the matrix of inner products $((x_i, x_j)) = \mathrm{diag}(a_1, \ldots, a_\ell)$ is diagonal. Then

$$\det(L) = \det((x_i, x_j)) = \prod_i a_i, \tag{2.1}$$

and the Hasse invariant is given by

$$h_F(L) = \prod_{i<j} (a_i, a_j)_F. \tag{2.2}$$

In case 2_-, if $Q \in M_n(D)$ is a skew-Hermitian matrix representing L, $\det L$ means the reduced norm of Q. In case 3, $E = F(\delta)$ where $\Delta = \delta^2 \in F^\times$.

Table 2.2

Invariants of ϵ-Hermitian forms ($F = \mathbb{R}$)

Case	invariants	
1_+ L symmetric bilinear	$\dim_F L$	$\mathrm{sig}\, L$
1_- L symplectic	$\dim_F L$	
2_+ L quaternion Hermitian	$\dim_D L$	$\mathrm{sig}\, L$
2_- L quaternion skew-Hermitian	$\dim_D L$	
3_+ L Hermitian	$\dim_E L$	$\mathrm{sig}\, L$
3_- L skew-Hermitian	$\dim_E L$	$\mathrm{sig}\, L$

Here $\mathrm{sig}\, L$ denotes the signature of L. In case 3_-, this is the usual signature of the Hermitian space obtained by scaling the skew Hermitian form on L by δ.

Using these facts, we can describe the map μ_V explicitly.

PROPOSITION 2.1: *Suppose that F is non-archimedean and let V be a fixed ϵ-Hermitian form with $m = \dim_D V$. Then the invariants of $\mu_V(L)$, the image of $L \in \mathrm{Herm}_\ell^\epsilon(D)$ under the map*

$$\mu_V\colon \mathrm{Herm}_\ell^\epsilon(D) \longrightarrow \mathrm{Sym}_{md\ell}(F), \qquad L \mapsto V \otimes_D L$$

are given as follows.

Case 1_+: $\dim \mu_V(L) = m\ell$, $\det(\mu_V(L)) = \det(V)^\ell \det(L)^m$ *and*

$$h_F(\mu_V(L)) = h_F(V)^\ell h_F(L)^m (\det(V), \det(L))_F^{m\ell-1} \times (-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}} (-1, \det(L))_F^{\frac{m(m-1)}{2}}.$$

Case 1_-: $\dim \mu_V(L) = m\ell$, $\det(\mu_V(L)) = 1$, $h_F(\mu_V(L)) = 1$.

Case 2_+: $\dim \mu_V(L) = 4m\ell$, $\det(\mu_V(L)) = 1$,

$$h_F(\mu_V(L)) = h_F(D)^{m\ell} = (-1)^{m\ell}(-1,-1)_F^{m\ell}.$$

Case 2_-: $\dim \mu_V(L) = 4m\ell$, $\det(\mu_V(L)) = 1$,

$$h_F(\mu_V(L)) = (-1)^{m\ell}(-1, \det(V))_F^\ell (-1, \det(L))_F^m (\det(V), \det(L))_F.$$

Case 3_+: $\dim \mu_V(L) = 2m\ell$, $\det(\mu_V(L)) = (-\Delta)^{m\ell}$, *and*

$$h_F(\mu_V(L)) = (-1, -\Delta)_F^{m\frac{\ell(\ell-1)}{2} + \ell\frac{m(m-1)}{2}} (\Delta, \det(V))_F^\ell (\Delta, \det(L))_F^m.$$

Case 3_-: $\dim \mu_V(L) = 2m\ell$, $\det(\mu_V(L)) = (-\Delta)^{m\ell}$, *and*

$$h_F(\mu_V(L)) = (-1, -\Delta)_F^{m\frac{\ell(\ell-1)}{2} + \ell\frac{m(m-1)}{2}} (\Delta, \delta^m \det(V))_F^\ell (\Delta, \delta^\ell \det(L))_F^m.$$

Recall that, in case 3, $D = E = F(\delta)$, with $\delta^2 = \Delta$. The notational conventions about determinants are explained in the proof.

The analogous result in the archimedean case is very easy:

PROPOSITION 2.2: *Suppose that $F = \mathbb{R}$. In cases 1_+, 2_+, and 3_+ let $\mathrm{sig}(V) = (p,q)$ where $p + q = \dim_D(V)$ and let $\mathrm{sig}(L) = (r,s)$ where $r + s = \dim_D(L)$. In case 3_-, fix a choice of δ and let (p,q) (resp. (r,s)) be the signature of the Hermitian space obtained by multiplying the form on V (resp. L) by δ. Then the invariants of the form $\mu_V(L)$ are given by:*

Case 1_+: *$\dim \mu_V(L) = m\ell$ and $\mathrm{sig}(\mu_V(L)) = (pr+qs, ps+qr)$*
Case 1_-: *$\dim \mu_V(L) = m\ell$ and $\mathrm{sig}(\mu_V(L)) = (\frac{m\ell}{2}, \frac{m\ell}{2})$*
Case 2_+: *$\dim \mu_V(L) = 4m\ell$ and $\mathrm{sig}(\mu_V(L)) = (4(pr+qs), 4(ps+qr))$*
Case 2_-: *$\dim \mu_V(L) = 4m\ell$ and $\mathrm{sig}(\mu_V(L)) = (2m\ell, 2m\ell)$*
Case 3_+: *$\dim \mu_V(L) = 2m\ell$ and $\mathrm{sig}(\mu_V(L)) = (2(pr+qs), 2(ps+qr))$*
Case 3_-: *$\dim \mu_V(L) = 2m\ell$ and $\mathrm{sig}(\mu_V(L)) = (2(pr+qs), 2(ps+qr))$*
Note that in this last case the result is independent of the choice of δ.

The proofs of these results are an easy case by case calculation, which will occupy the remainder of this section.

Proof of Proposition 2.1:

CASE 1_+: It is clear that $\dim_F V \otimes_F L = m\ell$ and $\det(V\otimes_F L) = \det(V)^\ell \det(L)^m$. On the other hand, choosing bases which diagonalize both forms and calculating, we obtain:

$$(2.3)\quad h_F(V\otimes_F L) = h_F(V)^\ell h_F(L)^m (\det(V), \det(L))_F^{m\ell-1} \times (-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}} (-1, \det(L))_F^{\frac{m(m-1)}{2}}.$$

CASE 1_-: In this case it is clear that $V\otimes_F L$ is a split quadratic space of dimension $m\ell$. Thus it has deteminant $(-1)^{m\ell} = 1$ and Hasse invariant

$$(-1,-1)_F^{\frac{m\ell(m\ell-1)}{2}} = 1,$$

since m and ℓ are both even.

CASE 2_+: Now $\dim_F(V\otimes_D L) = 4m\ell$. There exists a D-basis for V for which the form has matrix $\mathrm{diag}[a_1,\ldots,a_m]$ with $a_i \in F^\times$, Thus

$$(2.4)\qquad V\otimes_D L \simeq a_1\cdot L + \cdots + a_m \cdot L.$$

Also there exists a D-basis for L for which the form has matrix $\mathrm{diag}[b_1,\ldots,b_\ell]$ and hence

$$(2.5)\qquad L \simeq b_1 \cdot D + \cdots + b_\ell \cdot D$$

where D has the standard Hermitian form $(x,y)_D = xy^\tau$. The invariants of the associated quadratic form $(x,y) = \frac{1}{2}\mathrm{tr}(x,y)_D$ are $\dim_F(D) = 4$, $\det(D) = 1(\mod F^{\times,2})$ and $h_F(D) = -(-1,-1)_F$. Note that $h_F(b\cdot D) = h_F(D)$, and $h_F(L) = h_F(D)^\ell$. Thus we obtain $\det(V\otimes_D L) = (\det(V))^{4\ell}(\det(L))^{4m} = 1$ and $h_F(V\otimes_D L) = h_F(D)^{m\ell}$.

CASE 2_-: In this case we may choose D-bases for V and L for which the skew-Hermitian forms have matrices $\mathrm{diag}[\alpha_1, \ldots \alpha_m]$ and $\mathrm{diag}[\beta_1, \ldots, \beta_\ell]$ respectively. Here α_i and $\beta_j \in D^\times$ satisfy $\alpha_i^\tau = -\alpha_i$ and $\beta_j^\tau = -\beta_j$. For α of this type, let D_α denote the 1-dimensional left D vector space with skew Hermitian form with matrix α, i.e., $(x,y)_{D_\alpha} = x\alpha y^\tau$. Similarly, let ${}_\alpha D$ denote the 1-dimensional right D vector space with skew Hermitian form with matrix α, i.e., $(x,y)_{{}_\alpha D} = x^\tau \alpha y$. Then ${}_\alpha D \otimes_D D_\beta$ is a 4 dimensional vector space over F with symmetric F-bilinear form given by

$$(x_1 \otimes y_1, x_2 \otimes y_2) = \frac{1}{2}\mathrm{tr}\left((x_1^\tau \alpha x_2)(y_1 \beta y_2^\tau)^\tau\right). \tag{2.6}$$

LEMMA 2.3: *Let $a = \nu(\alpha)$ and $b = \nu(\beta)$ be the reduced norms of α and β. Then*

$$\det({}_\alpha D \otimes_D D_\beta) = 1$$

and

$$h_F({}_\alpha D \otimes_D D_\beta) = (a,b)_F h_F(D).$$

Using this we obtain $\det(V \otimes_D L) = 1$ and

$$h_F(V \otimes_D L) = (-1)^{m\ell}(-1, \det(V))_F^\ell\,(-1, \det(L))_F^m\,(\det(V), \det(L))_F. \tag{2.7}$$

Here $\det(V) = \nu(\alpha_1\alpha_2 \ldots \alpha_m)$ if α_i are as above, and similarly for $\det(L)$.

CASE 3_+: Let $\delta = -\delta^\tau$ be as above with $\Delta = \delta^2$. First we observe:

LEMMA 2.4: *If Q is an Hermitian space of dimension q over E and if RQ is the underlying $2q$ dimensional F vector space with quadratic form $\frac{1}{2}\mathrm{tr}(\ ,\ )$, then*

$$\det(RQ) = (-\Delta)^q$$

and

$$h_F(RQ) = (-1, -\Delta)_F^{\frac{q(q-1)}{2}}(\Delta, \det(Q))_F.$$

Now choose an E-basis for V for which the Hermitian form has matrix

$$\mathrm{diag}[a_1, \ldots, a_m],$$

so that

$$V \otimes_E L \simeq a_1 \cdot L + \cdots + a_m \cdot L. \tag{2.8}$$

Applying Lemma 2.4 and the properties of the Hasse invariant we obtain:

$$\det(V \otimes_E L) = (-\Delta)^{m\ell} \tag{2.9}$$

and

$$h_F(V \otimes_E L) = (-1, -\Delta)_F^{m\frac{\ell(\ell-1)}{2}+\ell\frac{m(m-1)}{2}} (\Delta, \det(V))_F^{\ell} (\Delta, \det(L))_F^{m}. \tag{2.10}$$

CASE 3_-: This case may be reduced to the previous by multiplying the forms by δ. If V' and L' are the resulting Hermitian forms, we find that $\det(V \otimes_E L) = \det(V' \otimes_E L')$ in $F^\times/F^{\times,2}$, and $h_F(V \otimes_E L) = h_F(V' \otimes_E L')$. Thus $R_V L$ and $R_{V'} L'$ are isometric as quadratic forms. The formulas of case 3_+ can then be applied. ∎

Proof of Proposition 2.2: Easy and omitted. ∎

3. Explicit trivializations

In this section we use the results of the previous two sections to give an explicit trivialization of the cocycle $(\iota_V)^* c_{\mathbb{Y}}$ where $Y \in \Omega(W)$ and $\mathbb{Y} = R_V Y \in \Omega(\mathbb{W})$.

THEOREM 3.1: *Let W and V be as in the introduction, with $\dim_D V = m$. Fix $Y \in \Omega(W)$, and let $\mathbb{Y} = R_V Y$. For the fixed additive character ψ, of F let $\eta = \frac{1}{2}\psi$, so that $\eta(x) = \psi(\frac{1}{2}x)$. In case 3, choose a character ξ of $E^\times$ whose restriction to $F^\times$ is the $\epsilon_{E/F}^m$, where $\epsilon_{E/F}(x) = (x, \Delta)_F$ is the quadratic character for the extension E/F. For $g \in P\tau_j P \subset G = G(W)$, the j-th cell, let $x(g)$ be as in (1.14) and Corollary 1.5 and let*

$$\beta_V(g) = \begin{cases} \gamma_F(x(g),\eta)^{-m}\,(x(g),\det(V))_F\,\gamma_F(\eta\circ V)^{-j} & \text{in case } 1_+ \\ 1 & \text{in case } 1_- \\ (-1)^{mj} & \text{in case } 2_+ \\ ((-1)^m\det(V),x(g))_F\,(-1,\det(V))_F^j\,h_F(D)^{mj} & \text{in case } 2_- \\ \xi(x(g))\,\gamma_F(\eta\circ RV)^{-j} & \text{in case } 3_+ \\ \xi(x(g))\xi(\delta)^j\gamma_F(\eta\circ RV')^{-j} & \text{in case } 3_- \end{cases}$$

Here, in case 3, RV is as in Lemma 2.4, and

$$\gamma_F(\eta\circ RV) = (\Delta,\det(V))_F\gamma_F(-\Delta,\eta)^m\gamma_F(-1,\eta)^{-m}.$$

Also, in case 3_-, V' denotes the Hermitian form obtained by scaling the skew Hermitian form on V by δ. Then, excluding the case 1_+ with m odd,

$$c_{\mathbb{Y}}(\iota_V(g_1), \iota_V(g_2)) = \beta_V(g_1 g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1}.$$

More precisely, in case 1_+,

$$c_{\mathbb{Y}}(\iota_V(g_1), \iota_V(g_2))\, \beta_V(g_1 g_2)^{-1}\beta_V(g_1)\beta_V(g_2)$$

is the m-th power of Rao's normalized (i.e., μ_2-valued) cocycle

$$c^0_{\mathbb{Y}}(g_1, g_2) = (x(g_1 g_2), -x(g_1)x(g_2))_F\, (x(g_1), x(g_2))_F \times (-1,-1)_F^{\frac{t(t-1)}{2}}\, ((-1)^t, \det(L_D))_F\, h_F(L_D),$$

and hence is trivial if and only if m is even. Here t is as in Proposition 1.6. Note that this cocycle is independent of V and ψ and, when m is even,

$$\beta_V(g) = \chi_V(x(g))\, \gamma_F(\eta \circ V)^{-j}$$

where $g \in P\tau_j P$ and

$$\chi_V(x) = (x, (-1)^{\frac{m}{2}} \det(V))_F,$$

a well known result.

Remark: J. Adams pointed out that the μ_2 valued cocycle given by Rao in his Theorem 5.3 should have the factor $(-1,-1)^{\frac{l(l-1)}{2}}$ in place of $(-1,-1)^{\frac{l(l+1)}{2}}$. Note that Rao's l is our t. Adams also pointed out an analogous error in our original version of Theorem 3.1!

Remark: Case 3 was also considered by Gelbart and Rogawski [4], section 3, who emphasized the importance of the dependence of the splitting on the choice of ξ.

COROLLARY 3.2: *Suppose that $F = \mathbb{R}$, and let $\psi(x) = e(x) = e^{2\pi i x}$. For $g \in P\tau_j P \subset G = G(W)$, let*

$$\beta_V(g) = \begin{cases} \gamma_{\mathbb{R}}(x(g), \psi)^{-m}(x(g), \det V)_{\mathbb{R}}\gamma(\psi \circ V)^{-j} & \text{in case } 1_+ \\ 1 & \text{in case } 1_- \\ (-1)^{mj} & \text{in case } 2_+ \\ 1 & \text{in case } 2_- \\ \xi(x(g))\, i^{-j(p-q)} & \text{in case } 3_+ \\ \xi(x(g))\, \xi(i)^j\, i^{-j(p-q)} & \text{in case } 3_- \end{cases}$$

Here, in case 3, ξ is any character of $\mathbb{C}^\times$ which extends the quadratic character $\chi_{\mathbb{C}}(x) = \mathrm{sgn}(x)^m$ of $\mathbb{R}^\times$. Then, excluding the case 1_+ with m odd,

$$c_{\mathbb{Y}}(\iota_V(g_1), \iota_V(g_2)) = \beta_V(g_1g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1}.$$

Remark: For $\psi(x) = e(x) = e^{2\pi i x}$, we have

$$\gamma_{\mathbb{R}}(a\cdot\psi) = e\big(\frac{\mathrm{sgn}(a)}{8}\big). \tag{3.1}$$

Recall that, in case 1_+, if V has matrix $((x_i, x_j)) = \mathrm{diag}(a_1, \ldots, a_m)$ for some basis $\{x_i\}$, then

$$\gamma_{\mathbb{R}}(\psi\circ V) = \gamma_{\mathbb{R}}(a_1\psi)\cdots\gamma_{\mathbb{R}}(a_n\psi) = e\big(\frac{1}{8}\big)^{p-q}, \tag{3.2}$$

since V has signature (p, q). Similarly,

$$\gamma_{\mathbb{R}}(a,\psi) = \frac{\gamma_{\mathbb{R}}(a\psi)}{\gamma_{\mathbb{R}}(\psi)} = \begin{cases} 1 & \text{if } a > 0 \\ -i & \text{if } a < 0. \end{cases} \tag{3.3}$$

Thus, in case 1_+, we have

$$\beta_V(g) = \begin{cases} e\big(\frac{1}{8}\big)^{-j(p-q)} & \text{if } x(g) > 0 \\ i^{p-q}\, e\big(\frac{1}{8}\big)^{-j(p-q)} & \text{if } x(g) < 0 \end{cases} \tag{3.4}$$

Corollary 3.3: *With the notation of Theorem 3.1, and excluding the case 1_+ with m odd, the map*

$$\tilde{\iota}_V(g) = (\iota_V(g), \beta_V(g))$$

defines a splitting of the restriction to G of the metaplectic cover:

$$\begin{array}{ccc} & & \mathrm{Mp}(\mathbf{W}) \simeq \mathrm{Sp}(\mathbf{W})\times\mathbb{C}^1 \\ & \overset{\tilde{\iota}_V}{\nearrow} & \downarrow \\ G & \xrightarrow{\iota_V} & \mathrm{Sp}(\mathbf{W}) \end{array}$$

The following relation will be useful in the proof of Theorem 3.1

Lemma 3.4: *If L, $(\ ,\ )$ is an inner product space over F which has a basis for which the form has matrix $\mathrm{diag}[a_1, \ldots, a_\ell]$, then, for any non-trivial additive character η, the Weil invariant is given by*

$$\gamma_F(\eta\circ L) = \gamma_F(\det(L), \eta)\,\gamma_F(\eta)^\ell\, h_F(L).$$

This expression depends only on the isometry class of L.

Proof:

$$
\begin{aligned}
\gamma_F(\eta \circ L) &= \gamma_F(a_1\eta)\dots\gamma_F(a_\ell\eta) \\
&= \gamma_F(a_1,\eta)\dots\gamma_F(a_\ell,\eta)\gamma_F(\eta)^\ell \\
&= \gamma_F(\det(L),\eta)\gamma_F(\eta)^\ell \prod_{i<j}(a_i,a_j)_F \\
&= \gamma_F(\det(L),\eta)\gamma_F(\eta)^\ell h_F(L).
\end{aligned} \tag{3.5}
$$

∎

Proof of Theorem 3.1: Once again we proceed case by case. As in the previous section we let

$$
\begin{aligned}
m &= \dim_D(V) \\
L_D &= L_D(Y, Yg_2^{-1}, Yg_1) \qquad \text{for } g_1 \text{ and } g_2 \in G \\
\ell &= \dim_D(L_D) \\
L &= \mu_V L_D, \\
\eta &= \frac{1}{2}\psi
\end{aligned} \tag{3.6}
$$

so that, in all cases, the cocycle in question is given by

$$
\begin{aligned}
\iota^* c_{\mathbb{Y}}(g_1,g_2) &= \gamma_F(\eta \circ L) \\
&= \gamma_F(\det(L),\eta)\gamma_F(\eta)^\ell h_F(L).
\end{aligned} \tag{3.7}
$$

The idea now is to calculate the quantities in this last expression in terms of the invariants of V and of L_D, as in Propositions 2.1 and 2.2. The notation of Propositions 1.3 and 1.6 will also be retained.

CASE 1_+: This is, in some ways, the most complicated case. By Proposition 1.6, we have

$$
\ell = \dim(L_D) = j_1 + j_2 - j - 2t \tag{3.8}
$$

and

$$
\det(L_D) = x(g_1g_2)x(g_1)x(g_2)(-1)^t. \tag{3.9}
$$

Thus

$$
\begin{aligned}
\gamma_F(\det(L),\eta) &= (\det(V),\det(L_D))_F^{m\ell}\gamma_F(\det(V)^\ell,\eta)\gamma_F(\det(L_D)^m,\eta) \\
&= (\det(V),\det(L_D))_F^{m\ell}(-1,\det(V))_F^{\frac{\ell(\ell-1)}{2}}\gamma_F(\det(V),\eta)^\ell \\
&\quad \times (-1,\det(L_D))_F^{\frac{m(m-1)}{2}}\gamma_F(\det(L_D),\eta)^m.
\end{aligned} \tag{3.10}
$$

and, as in section 2,

$$(3.11)\quad h_F(L) = h_F(V)^{\ell} h_F(L_D)^m (\det(V), \det(L_D))_F^{m\ell-1} \times (-1, \det(V))_F^{\frac{\ell(\ell-1)}{2}} (-1, \det(L_D))_F^{\frac{m(m-1)}{2}}.$$

This gives

$$(3.12)\quad \gamma_F(\eta \circ L) = h_F(V)^{\ell} h_F(L_D)^m (\det(V), \det(L_D))_F \times \gamma_F(\det(V), \eta)^{\ell} \gamma_F(\det(L_D), \eta)^m \gamma_F(\eta)^{m\ell}$$

Now set

$$(3.13)\quad \beta_0(g) = h_F(V)^j \, (\det(V), x(g))_F \, \gamma_F(\det(V), \eta)^{-j},$$

where $g \in P\tau_j P$. Then we have

$$(3.14)\quad \gamma_F(\eta \circ L) = \beta_0(g_1 g_2)\beta_0(g_1)^{-1}\beta_0(g_2)^{-1} \left[h_F(L_D)\gamma_F(\det(L_D), \eta)\gamma_F(\eta)^{\ell}\right]^m,$$

where we have used the fact that $(\det(V), (-1)^t)_F \gamma_F(\det(V), \eta)^{-2t} = 1$. Thus we have reduced our cocycle to an m-th power. On the other hand,

$$(3.15)\quad \begin{aligned} \gamma_F(\det(L_D), \eta) &= \left(x(g_1 g_2), -(-1)^t x(g_1) x(g_2)\right)_F \left((-1)^t x(g_1), x(g_2)\right)_F \\ &\times ((-1)^t, x(g_1))_F \gamma_F(x(g_1 g_2), \eta)^{-1} \gamma_F(x(g_1), \eta)\, \gamma_F(x(g_2), \eta)\, \gamma_F((-1)^t, \eta). \end{aligned}$$

and

$$(3.16)\quad \gamma_F(\eta)^{\ell} = \gamma_F(\eta)^{j_1} \gamma_F(\eta)^{j_2} \gamma_F(\eta)^{-j} \gamma(-1, \eta)^t,$$

since $\gamma_F(\eta)^2 = \gamma_F(-1, \eta)^{-1}$. Thus, if we set

$$(3.17)\quad \beta_V(g) = \beta_0(g)\gamma_F(x(g), \eta)^{-m}\gamma_F(\eta)^{-mj},$$

where $g \in P\tau_j P$, we have

$$(3.18)\quad \begin{aligned} \gamma_F(\eta \circ L) =& \beta_V(g_1 g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1} \\ &\times \big[h_F(L_D)\,(x(g_1 g_2), -(-1)^t x(g_1) x(g_2))_F\,((-1)^t x(g_1), x(g_2))_F \\ &\qquad ((-1)^t, x(g_1))_F\big]^m \\ &\times \left[\gamma_F((-1)^t, \eta)\gamma(-1, \eta)^t\right]^m. \end{aligned}$$

Here

$$\gamma_F((-1)^t,\eta)\gamma_F(-1,\eta)^t = (-1,-1)_F^{\frac{t(t+1)}{2}}. \tag{3.19}$$

Thus, by (3.9), our cocycle is cohomologous to the m-th power of the cocycle

$$\begin{aligned} c^0_{\mathrm{Y}}(g_1,g_2) = (x(g_1g_2), -x(g_1)x(g_2))_F\,(x(g_1),x(g_2))_F \\ \times (-1,-1)_F^{\frac{t(t-1)}{2}}\,((-1)^t,\det(L_D))_F\,h_F(L_D), \end{aligned} \tag{3.20}$$

which is clearly of order 2. This is precisely Rao's normalized cocycle, and the derivation is substantially as in his paper. If m is even, we have given an explicit trivialization of $\gamma_F(\eta\circ L)$, as required.

CASE 1_-: In this case, the Weil index of L is identically 1, so that there is nothing to do.

CASE 2_+: Now we have

$$\begin{aligned} \gamma_F(\eta\circ L) &= \gamma_F(\eta)^{4m\ell}h_F(D)^{m\ell} \\ &= (-1,-1)_F^{m\ell}(-(-1,-1)_F)^{m\ell} \\ &= (-1)^{m\ell}. \end{aligned} \tag{3.21}$$

Recall that, as in section 2, $h_F(D) = -(-1,-1)_F$. We let

$$\beta_V(g) = (-1)^{mj} \tag{3.22}$$

where $g\in P\tau_jP$, and obtain

$$\gamma_F(\eta\circ L) = \beta_V(g_1g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1} \tag{3.23}$$

as required.

CASE 2_-: Note that we now have $x(g_1g_2) = x(g_1)x(g_2)\det(L_D)$, since $\epsilon=-1$. Then

(3.24)

$$\begin{aligned} &\gamma_F(\eta\circ L) \\ &\quad= (-1)^{m\ell}(-1,\det(V))_F^{\ell}\,(-1,\det(L_D))_F^{m}\,(\det(V),\det(L_D))_F\gamma_F(\eta)^{4m\ell} \\ &\quad= (-1)^{m\ell}(-1,\det(V))_F^{\ell}\,((-1)^m\det(V), x(g_1g_2)x(g_1)x(g_2))_F\,(-1,-1)_F^{m\ell}. \end{aligned}$$

Setting

$$\beta_V(g) = ((-1)^m\det(V), x(g))_F\,(-1,\det(V))_F^{j}\,h_F(D)^{mj}, \tag{3.25}$$

we obtain the required trivialization.

CASE 3_+: This is the most interesting case. We have

$$\gamma_F(\eta\circ L) = (-1,-\Delta)_F^{m\frac{\ell(\ell-1)}{2}+\ell\frac{m(m-1)}{2}}\,(\Delta,\det(V))_F^{\ell}\,(\Delta,\det(L_D))_F^{m} \times \gamma_F((-\Delta)^{m\ell},\eta)\,\gamma_F(\eta)^{2m\ell}. \tag{3.26}$$

Now we observe that

$$\gamma_F((-\Delta)^{m\ell},\eta)\,\gamma_F(\eta)^{2m\ell} = (-1,-\Delta)_F^{\frac{m\ell(m\ell-1)}{2}}\,\gamma_F(-\Delta,\eta)^{m\ell}\gamma(-1,\eta)^{-m\ell}. \tag{3.27}$$

Set

$$\beta_0(g) = (\Delta,\det(V))_F^{j}\,\gamma_F(-\Delta,\eta)^{-mj}\gamma_F(-1,\eta)^{mj}. \tag{3.28}$$

Note that

$$\beta_0(g) = \gamma_F(\eta\circ RV)^{-j} \tag{3.29}$$

where RV is as in Lemma 2.4. In fact,

$$\begin{aligned}\gamma_F(\eta\circ RV) &= \gamma_F(\det(RV),\eta)\gamma_F(\eta)^{2m}h_F(RV)\\ &= \gamma_F((-\Delta)^m,\eta)\gamma_F(-1,\eta)^{-m}(-1,-\Delta)_F^{\frac{m(m-1)}{2}}(\Delta,\det(V))_F\\ &= \gamma_F(-\Delta,\eta)^m\gamma_F(-1,\eta)^{-m}(\Delta,\det(V))_F,\end{aligned} \tag{3.30}$$

as claimed. Removing the coboundary of β_0 leaves

(3.31)

$$(-1,-\Delta)_F^{m\frac{\ell(\ell-1)}{2}+\frac{m\ell(m\ell-1)}{2}+\ell\frac{m(m-1)}{2}}\,(\Delta,\det(L_D))_F^{m}\,\gamma_F(-\Delta,\eta)^{-2mt}\gamma_F(-1,\eta)^{2mt}.$$

Here

$$(-1,-\Delta)_F^{m\frac{\ell(\ell-1)}{2}+\frac{m\ell(m\ell-1)}{2}+\ell\frac{m(m-1)}{2}} = 1.$$

while the last two factors reduce to

$$\gamma_F(-\Delta,\eta)^{-2mt}\gamma_F(-1,\eta)^{2mt} = (-1,\Delta)_F^{mt}, \tag{3.32}$$

and

$$(\Delta,\det(L_D))_F^{m} = (\Delta, x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1})_F^{m}\,(-1,\Delta)_F^{mt}. \tag{3.33}$$

Thus (3.31) becomes

$$(\Delta, x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1})_F^{m}. \tag{3.34}$$

Note that, in this expression, the quantity $x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1}$ lies in $F^\times$, while the individual factors need not. Now choose a character ξ of $E^\times$ which extends the quadratic character

$$\epsilon_{E/F}^m(x) = (x, \Delta)_F^m \tag{3.35}$$

of $F^\times$. Then

$$(\Delta, x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1})_F^m = \xi(x(g_1g_2))\xi(x(g_1))^{-1}\xi(x(g_2))^{-1}. \tag{3.36}$$

Setting $\beta_V(g) = \beta_0(g)\,\xi(x(g))$, we have

$$\gamma_F(\eta \circ L) = \beta_V(g_1g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1}, \tag{3.37}$$

as claimed! Note that this trivialization depends on the choice of ξ, and that any two such choices differ by a character μ of $E^\times$ which is trivial on $F^\times$. Such a character corresponds to a character $\tilde{\mu}$ of E^1, the subgroup of $E^\times$ of elements of norm 1 to F. In fact we have

LEMMA 3.5: *In all cases, if $g \in P\tau_jP \subset G = G(W)$, then*

$$\det(g) = \epsilon^j \frac{x(g)}{x(g)^\tau}.$$

Proof: Since $\det(\tau_j) = \epsilon^j$ it suffices to check this identity on $p \in P$. But

$$p = \begin{pmatrix} ({}^ta^\tau)^{-1} & b \\ 0 & a \end{pmatrix} \tag{3.38}$$

with $x(p) = \det(a) \in E^\times$, and

$$\det(p) = \frac{x(p)}{x(p)^\tau}, \tag{3.39}$$

as claimed. ∎

Thus, any two choices of β differ by a character $\tilde{\mu} \circ \det$, as expected by general principles.

CASE 3_-: As in section 2, let V' (resp. W') denote the Hermitian (resp. skew Hermitian) space which is obtained from V (resp. W) by scaling the form by δ. Note that $G = G(W) = G(W') = G'$, and that, if g_1 and $g_2 \in G$ have Leray invariant $L_D = L_D(Y, Yg_2^{-1}, Yg_1)$, then the Leray invariant defined with respect to W' is just $(L_D)'$, the Hermitian form obtained from L_D by scaling by δ. Thus, $R_V L_D = R_{V'}(L_D)'$ and the cocycles pulled back via ι_V and $\iota_{V'}$ coincide. Thus, if we take

$$\beta_V(g) = \xi(x'(g))^m \gamma_F(\eta \circ RV')^{-j}, \tag{3.40}$$

the cocycle is the coboundary of β_V. There are two subtle points however. First, the function $x'(g)$ here is Rao's function for $G(W')$; this is related to the Rao function for $G(W)$ as follows.

LEMMA 3.6: *If $g \in P\tau_j P \subset G(W)$, then, viewing $g \in G(W')$,*

$$x'(g) = (-\delta)^{-j} x(g).$$

This relation is due to the dependence of $x(g)$ on the choice of standard basis.

Proof: If $e_1, \ldots, e'_n$ is the standard basis of W used to define $x(g)$, then

$$e_1, \ldots, e_n, -\delta^{-1}e'_1, \ldots, -\delta^{-1}e'_n \tag{3.41}$$

is the corresponding standard basis for W'. In particular, if $\tau \in G(W)$ and $\tau' \in G(W')$ are defined as in section 1.2, with respect to these standard bases, then

$$\tau' = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & -\delta \end{pmatrix} \tau. \tag{3.42}$$

Thus $x'(\tau) = (-\delta)^{-n} x'(\tau') = (-\delta)^{-n}$ and similarly, $x'(\tau_S) = (-\delta)^{-|S|}$. Since for $p \in P$, $x'(p) = x(p)$, the Lemma follows. ∎

Thus, since $\xi(-\delta\delta) = \xi(N(\delta)) = 1$, we may write

$$\beta_V(g) = \xi(x(g))^m \xi(\delta)^{mj} \gamma_F(\eta \circ RV')^{-j}, \tag{3.43}$$

Now the space RV' also depends on the choice of δ, but it is easily checked that the quantity $\xi(\delta)^m \gamma_F(\eta \circ RV')^{-1}$ does not. This completes the proof of Theorem 3.1. ∎

Proof of Corollary 3.2: When $F = \mathbb{R}$, a slightly more direct proof of Theorem 3.1 can be given as follows. Recall that, if L is a diagonal quadratic form, then

$$\gamma_{\mathbb{R}}(\eta \circ L) = \gamma_{\mathbb{R}}(a_1\eta) \cdots \gamma_{\mathbb{R}}(a_n\eta) \tag{3.44}$$

and thus, if L has signature (P,Q), we have

$$\gamma_{\mathbb{R}}(\eta\circ L)=\gamma_{\mathbb{R}}(\eta)^{P-Q}. \tag{3.45}$$

Now let $L_D = L_D(Y, Yg_2^{-1}, Yg_1)$ and set $L = \mu_V(L_D)$, as before. Let $\ell = \dim_D(L_D)$, and let $(p,q) = \mathrm{sig}(V)$ and $(r,s) = \mathrm{sig}(L_D)$, as in Proposition 2.3. Also let $(P,Q) = \mathrm{sig}(L)$. We then obtain

$$P-Q=\begin{cases}(p-q)(r-s) & \text{in case } 1_+\\ 0 & \text{in case } 1_-\\ 4(p-q)(r-s) & \text{in case } 2_+\\ 0 & \text{in case } 2_-\\ 2(p-q)(r-s) & \text{in case } 3_+\\ 2(p-q)(r-s) & \text{in case } 3_-.\end{cases} \tag{3.46}$$

We also recall from Proposition 1.6 that

$$\det(L_D)=x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1}(-\epsilon)^t \tag{3.47}$$

and

$$\ell=j_1+j_2-j-2t. \tag{3.48}$$

In cases 1_- and 2_- there is nothing to prove. Note that in cases 1_+, 2_+, and 3_+,

$$\mathrm{sgn}(\det(L_D))=(-1)^s, \tag{3.49}$$

while in case 3_-, taking $\delta=i$,

$$\mathrm{sgn}(i^\ell\det(L_D))=(-1)^s. \tag{3.50}$$

Writing $r-s=\ell-2s$ and using the relations just recalled, we easily obtain:

$$\begin{aligned}\gamma(\eta)^{\alpha(p-q)(r-s)}&=\gamma(\eta)^{\alpha(p-q)(\ell-2s)}\\ &=\big(\gamma(\eta)^\alpha\big)^{j_1+j_2-j}\big(\gamma(-1,\eta)^\alpha\big)^{s+t},\end{aligned} \tag{3.51}$$

where $\alpha=1$ in case 1_+, 4 in case 2_+ and 2 in case $3_\pm$. Thus,

$$\gamma(\eta)^{\alpha(p-q)(r-s)}=\begin{cases}(-1)^{m(j_1+j_2-j)} & \text{in case } 2_+\\ \gamma_{\mathbb{R}}(-1,\eta)^{-(p-q)(j_1+j_2-j)}(-1)^{m(s+t)} & \text{in cases } 3_\pm \text{ and}\\ \gamma_{\mathbb{R}}(\eta)^{(p-q)(j_1+j_2-j)}\gamma_{\mathbb{R}}(-1,\eta)^{(p-q)(s+t)} & \text{in case } 1_+\end{cases} \tag{3.52}$$

In case 2_+ we are then done, taking $\beta_V(g) = (-1)^j$. In case 3_+ we have

$$(3.53) \qquad (-1)^{s+t} = (-1)^t \det(L_D) = x(g_1g_2)x(g_1)^{-1}x(g_2)^{-1},$$

while in case 3_-, for $\delta = i$, we get

$$(3.54) \qquad \begin{aligned}(-1)^{s+t} &= i^{\ell} \det(L_D)(-1)^t \\ &= i^{-j}x(g_1g_2)i^{j_1}x(g_1)^{-1}i^{j_2}x(g_2)^{-1}.\end{aligned}$$

Taking a character ξ of $\mathbb{C}^\times$ whose restriction to $\mathbb{R}^\times$ is $\epsilon_{\mathbb{C}/\mathbb{R}}^m$, we let

$$(3.55) \qquad \beta_V(g) = \begin{cases} \xi(x(g))\gamma_{\mathbb{R}}(-1,\eta)^{(p-q)j} & \text{in case } 3_+ \\ \xi(x(g))\xi(i)^{-j}\gamma_{\mathbb{R}}(-1,\eta)^{(p-q)j} & \text{in case } 3_-. \end{cases}$$

This is the expression claimed in Corollary 3.2.

Finally, in case 1_+ with m and hence $p-q$ even, we have the same calculation as in case 3_+, except that the quantities $x(g)$ are already in $\mathbb{R}^\times$. Thus

$$(3.56) \qquad \beta_V(g) = (x(g),-1)_{\mathbb{R}}^{\frac{p-q}{2}} \gamma_{\mathbb{R}}(-1,\eta)^{\frac{(p-q)j}{2}},$$

will trivialize the cocycle in this case. Finally, if m is odd in case 1_+, the coboundary of

$$(3.57) \qquad \beta_V(g) = \gamma_{\mathbb{R}}(x(g),\eta)^{-m}(x(g),-1)_{\mathbb{R}}^{q}\gamma_{\mathbb{R}}(\eta)^{-j(p-q)}$$

will reduce the cocycle to Rao's $c_{\mathbb{Y}}^0$, just as in the non-archimedean case. ∎

4. Other unitary groups, some examples

We now consider the case in which the space W is not split, i.e., does not allow a decomposition as a direct sum of maximal isotropic subspaces. In this case, a doubling proceedure suggested by Michael Harris provides a kind of reduction to the split case. The result is not quite as explicit as before, but should still be adequate for many purposes.

We retain the notation of the previous sections.

First we observe that in case 1_+, W is always split, while in case 1_-, i.e., $G = O(W)$, we may as well take $\mathbb{Y} = Y \otimes_D W$ for some polarization $V = X + Y$. Then the image $\iota_V(G)$ lies in $P_{\mathbb{Y}}$ and the cocycle $c_{\mathbb{Y}}$ is trivial on this group. Thus we need only discuss cases $2_{\pm}$ and 3_{+} (We now avail ourselves of the non-canonical identification of 3_{-} with 3_{+}).

Let $\bar{W}$ denote the space W with the negative $-\langle\ ,\ \rangle$ as ϵ-skew Hermitian form. The space $W+\bar{W}$ is then a split space. Also let $\bar{\mathbb{W}} = V \otimes_D \bar{W}$, etc. We let $G(W)$ act on $W+\bar{W}$ via its natural action on W extended **trivially** on $\bar{W}$. We thus obtain a commutative diagram:

$$\text{(4.1)} \qquad \begin{array}{ccc} G(W+\bar{W}) & \xrightarrow{\ \iota_V\ } & \mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}}) \\ {\scriptstyle i}\uparrow & & \uparrow \\ G(W) & \xrightarrow{\ \iota_V \times \{1\}\ } & \mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\bar{\mathbb{W}}) \end{array}$$

where the lower horizontal arrow is given by $g \mapsto (\iota_V(g), 1)$.

Now, if $\mathbb{Y} \in \Omega(\mathbb{W}+\bar{\mathbb{W}})$ is in the image of the map R_V from $\Omega(W+\bar{W})$, an explicit trivialization of the cocycle $\iota_V^* c_{\mathbb{Y}}$ on the group $G(W+\bar{W})$ is given in Theorem 3.1 above. We write

$$\text{(4.2)} \qquad \iota_V^* c_{\mathbb{Y}} = \partial \beta_V,$$

where we write

$$\text{(4.3)} \qquad \partial\beta_V(g_1, g_2) = \beta_V(g_1 g_2)\beta_V(g_1)^{-1}\beta_V(g_2)^{-1}.$$

On the other hand, if $\mathbb{Y}' \in \Omega(\mathbb{W}+\bar{\mathbb{W}})$ has the form $\mathbb{Y}' = \mathbb{Y}'\cap\mathbb{W}+\mathbb{Y}'\cap\bar{\mathbb{W}} = \mathbb{Y}'_1+\mathbb{Y}'_2$, then the pullback of $c_{\mathbb{Y}}$ to the group $\mathrm{Sp}(\mathbb{W}) \times \mathrm{Sp}(\bar{\mathbb{W}})$ is the product of $c_{\mathbb{Y}'_1}$ on the first factor and $c_{\mathbb{Y}'_2}$ on the second. The further pullback $\iota_V c_{\mathbb{Y}'_1}$ is the cocycle whose explicit trivialization we seek. But finally, the cocycles $c_{\mathbb{Y}}$ and $c_{\mathbb{Y}'}$ are cohomologous on $\mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}})$, i.e.,

$$\text{(4.4)} \qquad c_{\mathbb{Y}'} = c_{\mathbb{Y}} \cdot \partial\lambda$$

for some function λ: $\mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}}) \longrightarrow \mathbb{C}^1$. Combining these facts we obtain

$$\text{(4.5)} \qquad \iota_V c_{\mathbb{Y}'_1} = \partial\big(i^*\beta_V\big) \cdot \partial\big((\iota_V \circ i)^*\lambda\big) = \partial\big(i^*(\beta_V \cdot \iota_V^*\lambda)\big),$$

as desired. This proves the known result [5]:

PROPOSITION 4.1: *In all cases except 1_+ with $m = \dim V$ odd, there is a splitting $G(W) \longrightarrow \mathrm{Mp}(\mathbf{W})$ of the restriction of the metaplectic cover to $G(W)$.*

In order to make this splitting 'explicit', we should still try to determine β_V and λ as explicitly as possible on $G(W)$ for suitable choices of $\mathbb{Y}$ and $\mathbb{Y}'$. For convenience, we will only consider the 3_+. The other cases can be treated similarly.

The form (,) on V can be diagonalized; so we may write $V = \oplus_i V_i$ as an orthogonal direct sum of one dimensional spaces. The space $\mathbf{W} = V \otimes_D W = \oplus_i V_i \otimes_D W = \oplus_i \mathbf{W}_i$ is likewise decomposed, and if $\mathbb{Y}' \in \Omega(\mathbf{W})$ is compatible with this decomposition, i.e., $\mathbb{Y}' = \oplus_i \mathbb{Y}'_i \cap \mathbf{W}_i$, then the restriction to $G(W)$ of the associated cocycle $c_{\mathbb{Y}'}$ is a product of the cocycles $\iota^*_{V_i} c_{\mathbb{Y}'_i}$. Thus we may as well assume that $\dim_D V = 1$.

Let $w_1, \dots, w_n$ be an E-basis for W for which the skew–Hermitian form has matrix

$$\delta \mathbf{a} = \delta \cdot \mathrm{diag}(a_1, \dots, a_n), \tag{4.6}$$

with $a_i \in F^\times$. For $g \in U(W)$, write $w \cdot g = (x + \delta y) \cdot w$ where w is the column vector whose components are the w_i's and $x, y \in M_n(F)$. Note that then

$$(x + \delta y)\mathbf{a}^t(x - \delta y) = \mathbf{a}. \tag{4.7}$$

Also note that if $V = E$ with $(v_1, v_2) = b v_1^\tau v_2$ for some $b \in F^\times$, then $\mathbf{W} \simeq R_{E/F} W$ with

$$\langle\langle\ ,\ \rangle\rangle = \frac{1}{2} b \cdot \mathrm{tr}_{E/F}(\langle\ ,\ \rangle). \tag{4.8}$$

(The factor $\frac{1}{2}$ is the κ from (0.10).) Thus, for purposes of calculation, we may as well assume that $b = 1$, by absorbing this scalar into the a_i's. Let

$$\mathbb{Y}'_1 = \mathrm{span}_F\{w_1, \dots, w_n\} \in \Omega(\mathbf{W}). \tag{4.9}$$

We want to trivialize the pullback $\iota^* c_{\mathbb{Y}'_1}$ on $G(W)$, following the above procedure.

We let $\bar{w}_i \in \bar{W}$ be the corresponding basis of $\bar{W}$, and we take

$$Y = \mathrm{span}_E\{w_1 + \bar{w}_1, \dots, w_n + \bar{w}_n\} \in \Omega(W + \bar{W}), \tag{4.10}$$

and let

$$\mathbb{Y} = \mathrm{span}_F\{\delta(w_1+\bar{w}_1),\dots,\delta(w_n+\bar{w}_n), w_1+\bar{w}_1,\dots,w_n+\bar{w}_n\} \tag{4.11}$$

be its image in $\Omega(\mathbf{W}+\bar{\mathbf{W}})$. By Theorem 3.1, we know how to trivialize the pullback of $c_{\mathbb{Y}}$ to $G(W+\bar{W})$. Also let

$$\begin{aligned}\mathbb{Y}' = \mathbb{Y}'_1+\mathbb{Y}'_2 = \mathrm{span}_F\{&-(a_1\Delta)^{-1}w_1,\dots,-(a_n\Delta)^{-1}w_n,\\ &-(a_1\delta)^{-1}\bar{w}_1,\dots,-(a_n\delta)^{-1}\bar{w}_n\}.\end{aligned} \tag{4.12}$$

Here $\Delta = \delta^2$. Note that these maximal isotropic subspaces have the properties required for the doubling argument outlined above. Moreover, we have a complete polarization

$$\mathbf{W}+\bar{\mathbf{W}} = \mathbb{Y}'+\mathbb{Y}, \tag{4.13}$$

and the bases we have given comprise, together, a standard symplectic F-basis

$$\begin{aligned}&-(a_1\Delta)^{-1}w_1,\dots,-(a_n\Delta)^{-1}w_n,-(a_1\delta)^{-1}\bar{w}_1,\dots,-(a_n\delta)^{-1}\bar{w}_n,\\ &\delta(w_1+\bar{w}_1),\dots,\delta(w_n+\bar{w}_n),w_1+\bar{w}_1,\dots,w_n+\bar{w}_n,\end{aligned} \tag{4.14}$$

for $\mathbf{W}+\bar{\mathbf{W}}$. On the other hand, we take

$$(2a_1\delta)^{-1}(w_1-\bar{w}_1),\dots,(2a_n\delta)^{-1}(w_n-\bar{w}_n), w_1+\bar{w}_1,\dots,w_n+\bar{w}_n \tag{4.15}$$

as our standard E-basis for $W+\bar{W}$.

Now a straightforward calculation shows that the matrix for the the image of $g = x+\delta y \in G(W)$ in $G(W+\bar{W})$, for our chosen basis (4.15), is

$$h = \frac{1}{2}\begin{pmatrix} \mathbf{a}^{-1}(g+1)\mathbf{a} & (2\delta\mathbf{a})^{-1}(g-1) \\ (g-1)2\delta\mathbf{a} & g+1\end{pmatrix} \in G(W+\bar{W}), \tag{4.16}$$

where we abuse notation and write g for $x+\delta y$. Thus one factor of our 'splitting' will be the function $g \mapsto \beta_V(h)$ on $G(W)$.

The matrix for the image of $g = x+\delta y \in G(W)$ in $\mathrm{Sp}(\mathbf{W}+\bar{\mathbf{W}})$ for the basis (4.14) is

$$h' = \begin{pmatrix} \mathbf{a}^{-1}x\mathbf{a} & -\mathbf{a}^{-1}y\mathbf{a} & -(\Delta\mathbf{a})^{-1}y & 0 \\ 0 & 1 & 0 & 0 \\ -\Delta^2 y\mathbf{a} & \Delta(x-1)\mathbf{a} & x & 0 \\ -\Delta(x-1)\mathbf{a} & \Delta y\mathbf{a} & y & 1\end{pmatrix} \in \mathrm{Sp}(\mathbf{W}+\bar{\mathbf{W}}). \tag{4.17}$$

Note that the basis (4.14) used here is not compatible with the decomposition $\mathbb{W}+\bar{\mathbb{W}}$, and this effects the form of h'.

Since we are interested in splitting the restriction of the cocycle $c_{\mathbb{Y}_1'}$ to $G(W)$, we consider the basis

$$\delta w_1, \dots, \delta w_n, (\Delta a_1)^{-1} w_1, \dots, (\Delta a_n)^{-1} w_n \tag{4.18}$$

for $\mathbb{W} = \mathbb{X}_1' + \mathbb{Y}_1'$. The image in $\mathrm{Sp}(\mathbb{W})$ of $g = x + \delta y \in G(W)$ has matrix

$$\begin{pmatrix} x & \Delta^2 \mathbf{a} y \\ (\Delta \mathbf{a})^{-1} y & \mathbf{a}^{-1} x \mathbf{a} \end{pmatrix} \tag{4.19}$$

with respect to this basis.

Next we need to relate the cocycle $c_{\mathbb{Y}}$, whose pullback is explicitly split, to $c_{\mathbb{Y}'}$.

Lemma 4.2: *The 2-cocycles $c_{\mathbb{Y}}$ and $c_{\mathbb{Y}'}$ associated to* **any** *$\mathbb{Y}, \mathbb{Y}' \in \Omega(\mathbb{W}+\bar{\mathbb{W}})$ are related as follows: Fix $\alpha \in \mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}})$ such that $\mathbb{Y}' = \mathbb{Y}\alpha$. Then*

$$c_{\mathbb{Y}'}(g_1, g_2) = c_{\mathbb{Y}}(\alpha g_1 \alpha^{-1}, \alpha g_2 \alpha^{-1}).$$

More explicitly, if

$$\lambda(g) = c_{\mathbb{Y}}(\alpha, g\alpha^{-1}) c_{\mathbb{Y}}(g, \alpha^{-1}),$$

then

$$c_{\mathbb{Y}'}(g_1, g_2) = \lambda(g_1 g_2) \lambda(g_1)^{-1} \lambda(g_2)^{-1} c_{\mathbb{Y}}(g_1, g_2).$$

Finally, by Rao's formula for the cocycle,

$$\lambda(g) = \gamma_F(\eta \circ L(\mathbb{Y}, \mathbb{Y}' g^{-1}, \mathbb{Y}'))\, \gamma_F(\eta \circ L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y} g)).$$

Note that the final formula here is independent of the choice of α.

If we let $w = \tau_{2n} \in \mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}})$ as in (1.9)–(1.10), then, for our special choice of $\mathbb{Y}$ and $\mathbb{Y}'$, we have $\mathbb{Y}' = \mathbb{Y} w$ and, for any $h \in \mathrm{Sp}(\mathbb{W}+\bar{\mathbb{W}})$, $h^{-1} = w\,{}^t h w^{-1}$. Then we obtain the following.

Lemma 4.3: *For $g \in U(W)$ and for h' the associated matrix, given by (4.17) above, the function λ which relates the cocycles $c_{\mathbb{Y}}$ and $c_{\mathbb{Y}'}$ is given by*

$$\begin{aligned} \iota_V^* \lambda(g) &= \gamma_F(\eta \circ L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y} h')) \cdot \gamma_F(\eta \circ L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}^t h')) \\ &= \gamma_F(\det(L), \eta)\, \gamma_F(\eta)^{\ell}\, h_F(L)\, \gamma_F(\det(L'), \eta)\, \gamma_F(\eta)^{\ell'}\, h_F(L'). \end{aligned}$$

Here $L = L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}h')$ and $L' = L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}^t h')$ have dimensions ℓ and ℓ' respectively.

Here we have used Lemma 3.4 to express the Weil index of the Leray invariants in terms of their determinant, dimension and Hasse invariant.

Next we compute the quantities $\det(L)$,$\det(L')$, ℓ and ℓ' in general. If we write

$$h' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.20}$$

then the dimension r of the space

$$\begin{aligned} R &= \mathbb{Y} \cap \mathbb{Y}' + \mathbb{Y}' \cap \mathbb{Y}h' + \mathbb{Y}h'' \cap \mathbb{Y} \\ &= \mathbb{Y}' \cap \mathbb{Y}h' + \mathbb{Y}h' \cap \mathbb{Y} \end{aligned} \tag{4.21}$$

is given by

$$r = 2n - \mathrm{rk}(D) + 2n - \mathrm{rk}(C). \tag{4.22}$$

Note that the dimension of the image of the projection of $\mathbb{Y}h'$ to $\mathbb{Y}$ (resp. $\mathbb{Y}'$) is $\mathrm{rk}(D)$ (resp. to $\mathrm{rk}(C)$). Thus, we have

$$\ell = \dim(L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}h')) = 2n - r. \tag{4.23}$$

But $\mathrm{rk}(C) = 2\mathrm{rk}(g-1)$ and $\mathrm{rk}(D) = n + \mathrm{rk}(x)$. We may argue similarly for $L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}^t h')$.

Lemma 4.4:

$$\ell = \dim L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}h') = 2\mathrm{rk}(g-1) + \mathrm{rk}(x) - n$$

and

$$\ell' = \dim L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}^t h') = \mathrm{rk}(x) + \mathrm{rk}(y) - n.$$

Invoking Proposition 1.6, and recalling that $j(w) = 2n$ and $x(w) = 1$, we have

Lemma 4.5: *For both $L = L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}h')$ and $L' = L(\mathbb{Y}, \mathbb{Y}', \mathbb{Y}^t h')$ the quantity t defined in Proposition 1.6 is $n - \mathrm{rk}(x)$. Also,*

$$\det(L) = (-1)^{n-\mathrm{rk}(x)} x(h'w) x(h')^{-1},$$

and

$$\det(L') = (-1)^{n-\mathrm{rk}(x)} x({}^t h' w) x({}^t h')^{-1}.$$

Combining ingredients, we obtain the main result of this section.

PROPOSITION 4.6: *For a skew-Hermitian space W, $\langle,\rangle$, with E-basis*

$$\{w_1, \ldots, w_n\}$$

as in (4.6), let $\mathbf{W} = R_{E/F}W$ and

$$\langle\langle\, ,\, \rangle\rangle = \frac{1}{2} tr_{E/F}(\langle\, ,\, \rangle).$$

For $g = x + \delta y \in G(W)$, define

$$\begin{aligned} \iota\colon G(W) &\longrightarrow \mathrm{Sp}(\mathbf{W}) \\ g &\mapsto \begin{pmatrix} x & \Delta^2 \mathbf{a} y \\ (\Delta \mathbf{a})^{-1} y & \mathbf{a}^{-1} x \mathbf{a} \end{pmatrix}, \end{aligned}$$

as in (4.19), and let $h \in G(W + \bar{W})$ and $h' \in \mathrm{Sp}(\mathbf{W} + \bar{\mathbf{W}})$ be given by (4.16) and (4.17) respectively. Let

$$\mathbb{Y}_0 = \mathrm{span}_F\{w_1, \ldots, w_n\} \in \Omega(\mathbf{W})$$

(This is $\mathbb{Y}_1'$ above). Then

$$\iota^* c_{\mathbb{Y}_0} = \partial\mu$$

where:

$$\mu(g) = \beta(h)\lambda(h')$$

with $\beta(h) = \beta_V(h)$ given by Theorem 3.1 with $V = E$ and $(x, y) = x^\tau y$. Also,

$$\begin{aligned} \lambda(h') = \gamma_F\big((-1)^{n-\mathrm{rk}(x)} x(h'w) x(h')^{-1}, \eta\big) \cdot \gamma_F\big((-1)^{n-\mathrm{rk}(x)} x({}^t h'w) x({}^t h')^{-1}, \eta\big) \\ \times \gamma_F(-1, \eta)^{\mathrm{rk}(g-1)+\mathrm{rk}(x)-n} \gamma_F(\eta)^{\mathrm{rk}(y)}\, h_F(L)\, h_F(L'). \end{aligned}$$

Here L and L' are as in Lemma 4.3 above.

Now, it only remains to determine the Hasse invariants of L and L'. We will do this only in the simplest case $n = 1$. It would be interesting to find analogous formulas for these invariants in the general case.

First observe that Yh' is the two dimensional isotropic subspace spanned by the last two rows of h', i.e., by the rows of (C, D) where, since $n = 1$,

$$(C, D) = \begin{pmatrix} -(\Delta^2 \mathbf{a}) y & (\Delta \mathbf{a})(x-1) & x & 0 \\ -(\Delta \mathbf{a})(x-1) & (\Delta \mathbf{a}) y & y & 1 \end{pmatrix}. \tag{4.24}$$

Note that, if $x \neq 0$, then, setting $Y'' = Yh'$, we have $R = Y \cap Y'' = 0$. Also,

$$\det(C) = (\Delta \mathbf{a})^2 N_{E/F}(g-1), \tag{4.25}$$

where $g = x + \delta y \in E^1$. Thus, if $g \neq 1$, C is invertible. The space Yh' is then also spanned by the rows of $(1, C^{-1}D)$, i.e.,

$$Yh' = Y' \begin{pmatrix} 1 & C^{-1}D \\ 0 & 1 \end{pmatrix}, \tag{4.26}$$

so that, if $g \neq 1$ and $x \neq 0$, the Leray invariant is

$$L = L(Y, Y', Yh') = C^{-1}D = (\Delta \mathbf{a})^{-1} N_{E/F}(g-1)^{-1} \begin{pmatrix} y & 1-x \\ 1-x & -\Delta y \end{pmatrix}. \tag{4.27}$$

Diagonalizing, we have

$$L = (\Delta \mathbf{a})^{-1} N_{E/F}(g-1)^{-1} \begin{pmatrix} y & 0 \\ 0 & 2y^{-1}x(1-x) \end{pmatrix} \left[\begin{pmatrix} 1 & y^{-1}(1-x) \\ 0 & 1 \end{pmatrix} \right]. \tag{4.28}$$

Thus, up to isometry,

$$L \simeq \begin{pmatrix} 2\Delta \mathbf{a} y(1-x) & \\ & \Delta \mathbf{a} x y \end{pmatrix}. \tag{4.29}$$

Similarly, $Y'' := Y{}^t h'$ is spanned by the rows of $({}^tB, {}^tD)$, i.e., by the rows of

$$\begin{pmatrix} -(\Delta \mathbf{a})^{-1} y & 0 & x & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, if $y = 0$, then $Y'' = Y$, $R = Y$ and $L' = 0$, while, if $y \neq 0$, $R = Y'' \cap Y$ is one dimensional and is spanned by the second row. In this case, $\ell' = 1$, $L' = (-\Delta \mathbf{a}) y^{-1} x$ and $h_F(L') = 1$.

This gives:

LEMMA 4.7: *In case $n = 1$, assume that $g \neq 1$ and $x \neq 0$. Then, $\ell = 2$, $x(i(g,1)) = (g-1)\delta \mathbf{a}$,*

$$\det(L) = 2x(1-x) \in F^\times / F^{\times,2},$$

and

$$h_F(L) = (-2x(1-x), \Delta \mathbf{a} x y)_F.$$

Also, if $y \neq 0$, then $\ell' = 1$,

$$L' = \det(L') = -\Delta \mathbf{a}xy \in F^\times/F^{\times,2},$$

and $h_F(L') = 1$.

Proof: We just check the Hasse invariant of L

$$\begin{aligned} h_F(L) &= (2\Delta\mathbf{a}y(1-x), \Delta\mathbf{a}xy)_F \\ &= (2\Delta\mathbf{a}y(1-x), \Delta\mathbf{a}xy)_F\,(-\Delta\mathbf{a}xy, \Delta\mathbf{a}xy)_F \\ &= (-2x(1-x), \Delta\mathbf{a}xy)_F \end{aligned} \tag{4.30}$$

as claimed. ∎

Combining all of this information, we obtain

PROPOSITION 4.8: *Assume that $n = 1$. Then, for $g \neq 1$ and $xy \neq 0$,*

$$\beta(h) = \xi(\delta\mathbf{a}(g-1))\,\gamma_F(\Delta, \eta).$$

and

$$\lambda(h') = \gamma_F(2\Delta\mathbf{a}y(1-x), \eta)\,\gamma_F(\eta).$$

These combine to give

$$\mu(g) = \xi(\delta(g-1))\,\gamma_F(2\mathbf{a}y(1-x), \eta)\,(\Delta, -2y(1-x))_F\,\gamma_F(\eta).$$

Proof: Using the expression in Lemma 4.3, and the result of Lemma 4.7,

$$\begin{aligned} \lambda(h') &= \gamma_F(2x(1-x), \eta)\,\gamma_F(\eta)^2\,\big(-2x(1-x), \Delta\mathbf{a}xy\big)_F\,\gamma_F(-\Delta\mathbf{a}xy, \eta)\,\gamma_F(\eta) \\ &= \gamma_F(2x(1-x), \eta)\,\gamma_F(-1, \eta)^{-1}\gamma_F(-1, \eta)\,\gamma_F(\Delta\mathbf{a}xy, \eta)\,(-1, \Delta\mathbf{a}xy)_F \\ &\qquad \times \big(-2x(1-x), \Delta\mathbf{a}xy\big)_F\,\gamma_F(\eta) \\ &= \gamma_F(2\Delta\mathbf{a}y(1-x), \eta)\,\big(2x(1-x), \Delta\mathbf{a}xy\big)_F\,(-1, \Delta\mathbf{a}xy)_F \\ &\qquad \times \big(-2x(1-x), \Delta\mathbf{a}xy\big)_F\,\gamma_F(\eta) \\ &= \gamma_F(2\Delta\mathbf{a}y(1-x), \eta)\,\gamma_F(\eta). \end{aligned} \tag{4.31}$$

Combining this with

$$\begin{aligned} \beta(h) &= \xi(\delta\mathbf{a}(g-1))\,\gamma_F(-\Delta, \eta)^{-1}\gamma_F(-1, \eta) \\ &= \xi(\delta\mathbf{a}(g-1))\,\gamma_F(\Delta, \eta), \end{aligned}$$

from Theorem 3.1, we obtain

$$(4.32)\qquad \begin{aligned}\mu(g) &= \xi(\delta\mathbf{a}(g-1))\,\gamma_F(\Delta,\eta)\,\gamma_F\big(2\Delta\mathbf{a}y(1-x),\eta\big)\,\gamma_F(\eta)\\ &= (\Delta,\mathbf{a})_F\xi(\delta(g-1))\,\gamma_F\big(2\mathbf{a}y(1-x),\eta\big)\,\big(\Delta,2\Delta\mathbf{a}y(1-x)\big)_F\,\gamma_F(\eta)\\ &= \xi(\delta(g-1))\,\gamma_F(2\mathbf{a}y(1-x),\eta)\,(\Delta,-2y(1-x))_F\,\gamma_F(\eta). \qquad\blacksquare\end{aligned}$$

Next we check that this indeed yields a splitting of the restriction of the standard cocycle for $SL_2(F)$, [8], p. 348, to the image of $U(1) \simeq E^1$ for the embedding, from (4.19):

$$(4.33)\qquad g = x+\delta y \mapsto \begin{pmatrix} x & \Delta^2\mathbf{a}y \\ (\Delta\mathbf{a})^{-1}y & x\end{pmatrix}.$$

This verification is based on the method of S. Brocco [1].

For simplicity, we suppose that α, β, and $\gamma \in E^1$ with $\alpha\beta\gamma = 1$, and we write

$$(4.34)\qquad \alpha = \frac{a+\delta}{a-\delta} \qquad \beta = \frac{b+\delta}{b-\delta} \quad\text{and}\quad \gamma = \frac{c+\delta}{c-\delta},$$

with a, b, and $c \in F$. In particular, we suppose, for the moment, that none of α, β or γ is equal to 1. Then, for $\alpha = x+\delta y$, say,

$$(4.35)\qquad x = \frac{a^2+\Delta}{a^2-\Delta} \quad\text{and}\quad y = \frac{2a}{a^2-\Delta},$$

and similarly for α and γ. It is easy to check that the condition $\alpha\beta\gamma = 1$ implies that

$$(4.36)\qquad -\Delta = ab+bc+ca$$

and also that

$$(4.37)\qquad (a^2-\Delta)(b^2-\Delta)(c^2-\Delta) = (a+b)^2(b+c)^2(c+a)^2.$$

and

$$(4.38)\qquad (a-\delta)(b-\delta)(c-\delta) = -(a+b)(b+c)(c+a).$$

We must show that the quantity

$$(4.39)\qquad \mu(\gamma^{-1})\mu(\alpha)^{-1}\mu(\beta)^{-1}$$

coincides with the cocycle, obtained using (4.18),

$$
\begin{aligned}
(4.40)\qquad c(\alpha,\beta) &= \gamma_F\left(\left(\Delta\mathbf{a}\frac{2a}{a^2-\Delta}\right)\left(\Delta\mathbf{a}\frac{2b}{b^2-\Delta}\right)\left(\Delta\mathbf{a}\frac{-2c}{c^2-\Delta}\right),\eta\right)\gamma_F(\eta)\\
&= \gamma_F(-2\Delta\mathbf{a}\cdot abc)\,\gamma_F(\eta).
\end{aligned}
$$

Note that $\alpha\beta=\gamma^{-1}$ corresponds to $-c$.

Now we have, for $\alpha = x+\delta y$, as above,

$$
(4.41)\qquad \delta(\alpha-1)=\frac{2\Delta}{a-\delta} \quad\text{and}\quad 2y(1-x)=\frac{-8\Delta a}{(a^2-\Delta)^2}\equiv -2\Delta a \mod F^{\times,2}.
$$

Then

$$
\begin{aligned}
(4.42)\qquad \mu(\alpha) &= (-2,\Delta)_F\,\xi(a+\delta)\,\gamma_F(-2\Delta\mathbf{a}a,\eta)\,(\Delta,2\Delta a)_F\,\gamma_F(\eta)\\
&= (\Delta,a)_F\xi(a+\delta)\,\gamma_F(-2\Delta\mathbf{a}a,\eta)\,\gamma_F(\eta).
\end{aligned}
$$

It is easy to check that

$$
(4.43)\qquad \mu(\alpha^{-1})=\mu(\alpha)^{-1},
$$

and so (4.39) becomes

$$
\begin{aligned}
(4.44)\qquad \mu(\gamma^{-1})\mu(\alpha^{-1})\mu(\beta^{-1}) =&(\Delta,abc)_F\xi(a+\delta)^{-1}\xi(b+\delta)^{-1}\xi(c+\delta)^{-1}\\
&\times\gamma_F(2\Delta\mathbf{a}a,\eta)\,\gamma_F(2\Delta\mathbf{a}b,\eta)\,\gamma_F(2\Delta\mathbf{a}c,\eta)\,\gamma_F(\eta)^3\\
=&(\Delta,abc)_F\Big(\xi(a+\delta)\,\xi(b+\delta)\,\xi(c+\delta)\Big)^{-1}\\
&\times\gamma_F(2\Delta\mathbf{a}abc,\eta)\,\gamma_F(-1,\eta)^{-1}\gamma_F(\eta)\\
&\times(2\Delta\mathbf{a}a,2\Delta\mathbf{a}b)_F\,(ab,2\Delta\mathbf{a}c)_F\\
=&(\Delta,abc)_F\Big(\xi(a+\delta)\,\xi(b+\delta)\,\xi(c+\delta)\Big)^{-1}\\
&\times\gamma_F(2\Delta\mathbf{a}abc,\eta)\,\gamma_F(-1,\eta)^{-1}\gamma_F(\eta)\\
&\times(2\Delta\mathbf{a},-1)_F(a,b)_F(b,c)_F(c,a)_F.
\end{aligned}
$$

Next we apply the following identity:

Lemma 4.9 (S. Brocco [1]): *For α,β,and $\gamma\in E^1$ as above,*

$$
\xi(a+\delta)\,\xi(b+\delta)\,\xi(c+\delta)=(\Delta,abc)_F(-1,-abc)_F\,(a,b)_F\,(b,c)_F\,(c,a)_F.
$$

Proof: For covenience, we include the proof. Recall that $(a,b)_F = (-ab, a+b)_F$.

$$\begin{aligned}(a,b)_F(b,c)_F(c,a)_F &= (ab,c)_F(a,b)_F \\ &= (ab,c)_F(-ab,a+b)_F \\ &= (ab,c(a+b))_F(-1,a+b)_F \\ &= (-1,a+b)_F(-abc(a+b),c(a+b)+ab)_F \qquad (4.45) \\ &= (-1,a+b)_F(-abd(a+b),-\Delta)_F \\ &= (-1,-abc)_F(-abc(a+b),\Delta)_F \\ &= (-1,-abc)_F(-abd(a+b)(b+c)(c+a),\Delta)_F\end{aligned}$$

(by symmetry)

$$\begin{aligned}&= (-1,-abc)_F\left(\frac{abc}{(a-\delta)(b-\delta)(c-\delta)},\Delta\right)_F \\ &= (-1,-abc)_F(\Delta,abc)_F\,\xi(x+\delta)\xi(b+\delta)\xi(c+\delta).\end{aligned}$$ ∎

Using this in (4.44), we obtain:

$$\begin{aligned}&(\Delta,abc)_F\,(\Delta,abc)_F\,(-1,-abc)_F \\ &\qquad\times\gamma_F(2\Delta\mathbf{a}abc,\eta)\,\gamma_F(-1,\eta)\,(-1,-1)_F\,\gamma_F(\eta)(2\Delta\mathbf{a},-1)_F \\ &= (-1,2\Delta\mathbf{a}abd)_F\,\gamma_F(2\Delta\mathbf{a}abc,\eta)\,\gamma_F(-1,\eta)\,\gamma_F(\eta) \\ &= \gamma_F(-2\Delta\mathbf{a}abc,\eta)\,\gamma_F(\eta)!\end{aligned} \qquad (4.46)$$

This is the desired result in the case where α, β and γ satisfy the conditions of Proposition 4.8. The remaining special cases, e.g., where one of α, β and γ is ± 1, are left to the reader.

5. Schrödinger models

We now return to the situation of sections 1–3 and assume that W is a split ϵ-Hermitian form with a fixed complete polarization $W = X + Y$, and compatible standard D-basis $e_1, \ldots, e_n$ as in §1.2. We want to write down the operators for the Weil representation $\omega_V = \omega \circ \tilde{\iota}_V$ of $G = U(W)$ for the standard Schrödinger model [11], [8].

Note that, for the complete polarization $\mathbf{W} = \mathbb{X} + \mathbb{Y}$, we have $\mathbb{X} \simeq V^n$ (row vectors of length n). For any element

$$\sigma = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \mathrm{Sp}(\mathbf{W}),$$

Rao defines a unitary operator $r(\sigma)$ on $L^2(\mathbb{X})$. It is given explicitly as follows: Let $\mathbb{Y}_\sigma = \mathbb{Y}/\ker(\gamma)$ and let $d\mu_\sigma$ be a suitably normalized (cf. [8]) Haar measure on $\mathbb{Y}_\sigma$. Then for $\varphi \in S(\mathbb{X})$ and $x \in \mathbb{X}$,

$$r(\sigma)\varphi(x) = \int_{\mathbb{Y}_\sigma} f_\sigma(x+y)\,\varphi(x\alpha + y\gamma)\,d\mu_\sigma(y),$$

where

$$f_\sigma(x+y) = \psi\big(\frac{1}{2}\langle x\alpha, x\beta\rangle + \langle y\gamma, x\beta\rangle + \frac{1}{2}\langle y\gamma, y\delta\rangle\big).$$

In our case, if we identify both $\mathbb{X}$ and $\mathbb{Y}$ with V^n, as above, we have

$$\langle (x_1,y_1),(x_2,y_2)\rangle = \kappa\cdot \mathrm{tr}_D\mathrm{tr}\big((x_1,y_2) - \epsilon(y_1,x_2)\big),$$

where κ is as in (0.9). Thus we obtain, for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

$$f_{\iota_V(g)}(x+y) = \psi\bigg(\frac{1}{2}\kappa\cdot \mathrm{tr}_D\mathrm{tr}(xa,xb) + \kappa\cdot \mathrm{tr}_D\mathrm{tr}(yc,xb) + \frac{1}{2}\kappa\cdot\mathrm{tr}_D\mathrm{tr}(yc,yd)\bigg).$$

For β_V is the function defined in Theorem 3.1, we set

$$\begin{aligned}\omega_V(g)\varphi(x) &= \beta_V(g)\int_{V_g^n}\psi\bigg(\frac{1}{2}\kappa\cdot \mathrm{tr}_D\mathrm{tr}(xa,xb) + \kappa\cdot \mathrm{tr}_D\mathrm{tr}(yc,xb) + \frac{1}{2}\kappa\cdot\mathrm{tr}_D\mathrm{tr}(yc,yd)\bigg)\\ &\qquad\qquad \varphi(xa+yc)\,d\mu_g(y),\end{aligned}$$

and, excluding the case 1_+ with m odd, we obtain a smooth representation of G on $S(V^n)$.

In the case 1_+ with m odd, the operators $\omega_V(g)$ define a representation of the twofold cover $\mathrm{Sp}_n(F)\times\mu_2$ where the cocycle $c^0_{\mathbb{Y}}(g_1,g_2)$ is Rao's normalized cocycle, given in section 3 above.

Finally, we note that the image of the group $H = U(V)$ under the embedding ι_W lies in the Levi factor of the parabolic subgroup of $\mathrm{Sp}(\mathbb{W})$ which stabilizes $\mathbb{Y}$. Since Rao's cocycle for $\mathrm{Sp}(\mathbb{W})$ is trivial on this subgroup, there is a natural splitting

$$\begin{aligned} U(V) &\longrightarrow \mathrm{Mp}(\mathbb{W}) \simeq \mathrm{Sp}(\mathbb{W})\times\mathbb{C}^1\\ h &\longrightarrow (h,1),\end{aligned}$$

and the resulting action of the group $H = U(V)$ on $S(V^n)$ in this model is just the natural linear action

$$\omega(h)\varphi(x) = \varphi(h^{-1}x).$$

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